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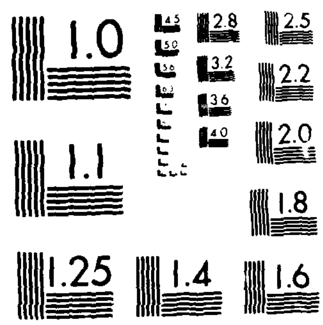

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Analytical Methods in Stochastic Control and Nonlinear Filtering

Final Report

ARO Contract: DAAL-03-86-C-0014

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Contents

Executive Summary

The focus of this report is on advanced tools for the analysis of nonlinear stochastic control and filtering systems.

In sections 1 and 2[1] we present a series of results on the analysis of certain classes of nonlinear filtering problems using comparatively simple bounding techniques. We consider both problems with small noise (large signal to noise ratios) and weakly nonlinear systems. We show that the optimal nonlinear filters can be well approximated by linear filters which are very easy to implement. Moreover, we provide sharp estimates of the degree of suboptimality involved in using the linear approximating filters.

In section 3[2] we consider the problem of managing the estimation of a (nonlinear) diffusion process by a system employing several sensors. The essential problem is to "schedule" the use of the sensor to optimize the estimate of a function of the state of the diffusion process. The solution is optained in terms of a system of quasi–variational inequalities in the space of solutions of certain Zakai equations.

In section 4 we provide a new proof of the minimum principle in stochastic optimal control theory for systems of partially observed diffusions. In section 5[3] we provide a concise analysis of the "conditional adjoint process" arising in the stochastic minimum principle for partially observed diffusion processes.

The sections may be read independently.

[1] The work in these sections is joint work by L. Saydy and G.L. Blankenship.
[2] The work in this section is joint work by J.S. Baras and A. Bensoussan.
[3] The work in sections 4 and 5 is joint work by J.S. Baras, R.J. Elliot and M. Kohlmann.

1 Optimal Stationary Behavior in Nonlinear Filtering Problems: A Bound Approach

1.1 Introduction

We consider the Ito stochastic model:

$$\begin{aligned} dx_t &= g(t, x_t)dt + \sigma(t)dw_t \\ dy_t &= h(t, x_t)dt + \rho(t)dv_t \\ x(0) &= x_0; \quad 0 \le t \le T \end{aligned} \tag{1}$$

where g, h, α and ρ are smooth functions of their arguments, $\{v_t\}, \{w_t\}$ are independent Wiener processes, x_0 a random variable independent of $\{v_t\}, \{w_t\}$.

Given this model one is interested in computing least squares estimates of functions of the signal x_t given $\sigma\{y_s, 0 \le s \le t\}$, the σ-algebra generated by the observations, i.e. quantities of the form $E[\phi(x_t)|\sigma\{y_s, 0 \le s \le t\}]$. In many applications this computation must be done recursively. This involves the conditional probability density $p^y(t, x)$ which satisfies a nonlinear stochastic partial differential equation, the Kushner-Stratonovich equation [16]. By considering an unnormalized version of p^y, the above problem can be reduced to the study of the Duncan-Mortenson-Zakai (DMZ) equation which is linear ([2]).

The filtering problem was completely solved in the context of finite dimensional linear Gaussian systems by Kalman and Bucy [17,18] in 1960-61, and the resulting Kalman filter (KF) has been widely applied. Apart from a few special cases [3,23], the nonlinear case is far more complicated; the evolution of the conditional statistics is, in general, an infinite dimensional system.

Although progress has been made using the DMZ equation, optimal algorithms are not generally available. Suboptimal filters are thus of interest. The performance of suboptimal designs, however derived, may be based on lower and upper bounds on the minimum mean square error (optimal MS-error) $p(t)$. This approach is used here to investigate the asymptotic behavior of a class of nonlinear filtering problems.

Two aspects are treated in detail:

1. the long time behavior, that is, the asymptotic behavior of the filter as $t \to \infty$ (this section; see also the paper [21]).
2. the asymptotic behavior as $\epsilon \to 0$, with ϵ a small parameter in the model (in the next section 2; see also the paper [22]).

To illustrate the ideas, consider the one-dimensional version of the model where g and h have continuous bounded derivatives, say

$$\underline{\alpha}(t) \leq g_x(t,x) \leq \bar{\alpha}(t) \tag{2}$$

$$\underline{\beta}(t) \leq h_x(t,x) \leq \bar{\beta}(t) \tag{3}$$

and let

$$\begin{aligned} p(t) &:= E[x_t - E(x_t|\mathcal{Y}_0^t)]^2 \\ p^*(t) &:= E(x_t - x_t^*)^2 \end{aligned} \tag{4}$$

where $\mathcal{Y}_0^t = \sigma\{y_s, 0 \leq s \leq t\}$ and x_t^* is given by:

$$\begin{aligned} dx_t^* &= g(t,x_t^*)dt + \frac{\beta(t)}{\rho^2(t)}u(t)[dy_t - h(t,x_t^*)dt]; \quad x^*(0) = 0 \\ \dot{u}(t) &= \sigma^2(t) + 2\bar{\alpha}(t)u(t) - \frac{\beta^2(t)}{\rho^2(t)}u^2(t); \quad u(0) = \sigma_0^2 \end{aligned} \tag{5}$$

$$(x_0 \sim \mathcal{N}(0,\sigma_0^2) \text{ assumed})$$

Clearly the BOF (bound optimal filter) (5) is readily implementable, with precomputable gain. It coincides with the Kalman filter if g and h are linear. In section 1.2 it is shown by applying results from [7,13] that the BOF is a "best bound" filter in the sense that the associated upper bound $u(t)$ of $p^*(t)$ is the tightest over a class of nonlinear Kalman-like filters and that $p(t)$ is bounded as follows:

$$0 \leq \ell(t) \leq p(t) \leq p^*(t) \leq u(t)$$

where $\ell(t)$ satisfies another Riccati equation.

In section 1.3 these bounds are used to address the long time behavior of asymptotically time invariant systems. In the particular case where

$$g(t,x) = ax + \lambda(t)f(t,x) \longrightarrow ax \text{ as } t \to \infty$$

and

$$h(t,x) = cx + \nu(t)k(t,x) \longrightarrow cx \text{ as } t \to \infty$$

it is shown that the BOF is asymptotically optimal in the sense that

$$\lim_{t\to\infty}(p^*(t) - p(t)) = 0$$

and that as far as the long time performance is concerned, the nonlinearities f and k can be ignored in the original model. In other words the "KF" and even the "SSKF" (steady state) formally designed for the underlying linear system are asymptotically optimal.

In section 1.4 examples with simulation results are given.

1.2 Lower and Upper bounds on the a priori Optimal MS-Error

Since the explicit solution of nonlinear filtering problems is impossible in general, one is naturally interested in suboptimal solutions, the performance of which may be avaluated using upper and lower bounds on the (unknown) optimal MS-error. In fact, the structural complexity which arises is also present at the level of performance testing in the sense that simple and tractable bounds are not generally available for suboptimal estimators unless one puts further restrictions on the type of nonlinearities considered.

Consider the one dimensional Ito stochastic differential equation

$$\begin{aligned} dx_t &= g(t,x_t)dt + \sigma(t)dw_t \\ dy_t &= h(t,x_t)dt + \rho(t)dv_t \\ x(0) &= x_0; \quad 0 \le t \le T \end{aligned} \tag{6}$$

$$x_0 \sim p_0(x), \quad Ex_0 = 0, \quad Ex_0x_0^t = \sigma_0^2$$

where $\{w_t\}$ and $\{v_t\}$ are independent standard Wiener processes, x_0 is a random variable (generally taken to be Gaussian) independent of $\{w_t\}$ and

$\{v_t\}$; g and h are such that (6) has a unique solution [1], differentiable with continuous partial derivatives. Given this model one is interested in finding bounds on the optimal MS-error:

$$p(t) = E[(x_t - E(x_t|\mathcal{Y}_0^t))^2] \tag{7}$$

where $\mathcal{Y}_0^t = \sigma\{y_s, 0 \leq s \leq t\}$ is the σ-algebra generated by the observations up to time t; i.e., find functions $\ell(t), u(t)$ such that:

$$0 \leq \ell(t) \leq p(t) \leq u(t) \tag{8}$$

In this section, existing results are applied to one dimensional systems for which the nonlinearities have bounded derivatives to obtain lower and upper bounds involving ordinary differential equations of the Riccati type. The upper bound is obtained in subsection 1.2.2 by considering a class of nonlinear, Kalman-like suboptimal filters. To each such filter is associated an upper bound on the corresponding mean square error (MSE) and the BOF (bound optimal filter) is defined as the one with the tightest upper bound. The latter is used in inequality (8).

1.2.1 Lower bound

The following additional assumptions make it possible to derive a simple, tractable lower bound in the one dimensional case:

$\mathcal{H}_1$: $|g_x(t,x) - \alpha(t)| \leq \Delta\alpha(t)$

$\mathcal{H}_2$: $|h_x(t,x) - \beta(t)| \leq \Delta\beta(t), \underline{\beta}(t) := \beta(t) - \Delta\beta(t) \geq 0$

We will denote this by:

$$g \in\prec [\alpha(t), \Delta\alpha(t)]$$

$$h \in\prec [\beta(t), \Delta\beta(t)]$$

Remark: The symbol Δ serves to exhibit the fact that $\Delta\alpha$ is a slope departure function.

Proposition 2-1:

Assume $\mathcal{H}_1, \mathcal{H}_2$ hold and let $p(t) := E(x_t - E(x_t|\mathcal{Y}_0^t))^2$; then $p(t)$ is lower bounded by $\ell(t)$, i.e., $0 \le \ell(t) \le p(t)$ where $\ell(t)$ satisfies the following Riccati equation:

$$\dot{\ell}(t) = \sigma^2(t) + 2\underline{\alpha}(t)\ell(t) - \frac{1}{\rho^2(t)}[\bar{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}(\Delta\alpha(t))^2]\ell^2(t) \qquad (9)$$

$$\ell(0) = \sigma_0^2$$

with the notation: $\bar{\alpha} = \alpha + \Delta\alpha, \underline{\alpha} = \alpha - \Delta\alpha$.

Remark: The above proposition says that the optimal MS-error $p(t)$ corresponding to the nonlinear filtering problem (6) is lower bounded by the optimal MS-error corresponding to the following Kalman filtering problem:

$$\begin{aligned} dz_t &= \underline{\alpha}(t)z_t dt + \sigma(t)dw_t \\ dy_t' &= \bar{\beta}(t)z_t dt + \rho'(t)dv_t \\ \rho'(t) &= \frac{\rho(t)}{\left(\frac{1+4\rho^2(t)}{\sigma^2(t)}\frac{\Delta\alpha^2(t)}{\bar{\beta}^2(t)}\right)^{\frac{1}{2}}} \end{aligned}$$

It is easily seen (e.g., [16]) that:

$$E[z_t - E(z_t|\sigma\{y_s' : 0 \le s \le t\})]^2 = \ell(t)$$

Proof:

Using the Bobrovsky-Zakai lower bound [7] we get that $\ell(t) \le p(t)$ where

$$\dot{\ell}(t) = \sigma^2(t) + 2a(t)\ell(t) - \frac{c^2(t)}{\rho^2(t)}L^2(t), \qquad L(0) = \sigma_0^2$$

$$a(t) = Eg_x(t, x_t); \qquad c^2(t) = Eh_x^2(t, x_t) + \frac{\rho^2(t)}{\sigma^2(t)}\mathrm{var}(g_x(t, x_t))$$

Thus, $\ell(t)$ satisfies a Riccati equation, the coefficients of which are unknown in general.

Clearly $\mathcal{H}_1$ implies: $\underline{\alpha}(t) \le g_x \le \bar{\alpha}(t)$ a.s., and hence , $\underline{\alpha}(t) \le a(t) \le \bar{\alpha}(t)$. Thus,

$$|g_x(t, x_t) - a(t)| \le 2\Delta\alpha(t) \text{ a.s.}$$

and

$$\mathrm{var} g_x(t, x_t) \leq 4(\Delta\alpha(t))^2$$

Similarly $\mathcal{H}_2$ implies: $0 \leq \underline{\beta}(t) \leq h_x(t, x_t) \leq \bar{\beta}(t)$ hence $E h_x^2(t, x_t) \leq \bar{\beta}^2(t)$. Therefore:

$$c^2(t) \leq \bar{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}(\Delta\alpha(t))^2$$

Since $L(t)$ satisfies a Riccati equation with strictly positive initial condition, then $L(t) > 0$ [11] and the right hand side of $\dot{\ell}(t)$ is hence greater than

$$\sigma^2(t) + 2\underline{\alpha}(t)\ell(t) - \frac{1}{\rho^2(t)}\left[\bar{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}(\Delta\alpha(t))^2\right] L^2$$

By the comparison theorem (see appendix[4]) we obtain: $\ell(t) \leq L(t)$

1.2.2 Upper bound and bound optimal filter (BOF)

Let x_t and y_t be as in (1) and assume that

$$\mathcal{H}_1 : g_x(t,x) \text{ is } \textit{continuous} \text{ and } g_x(t,x) \leq \bar{\alpha}(t)$$

$$\mathcal{H}_2 : h_x(t,x) \text{ is continuous and } h_x(t,x) \geq \underline{\beta}(t) \geq 0$$

Proposition 2-2:

The optimal MS-error $p(t)$ is upper bounded by $u(t)$ where $u(t)$ satisfies the Riccati equation:

$$\dot{u}(t) = \sigma^2(t) + 2\bar{\alpha}(t)u(t) - \frac{\underline{\beta}^2(t)}{\rho^2(t)}u^2(t) \tag{10}$$

$$u(0) = \sigma_0^2$$

Remark: This says that the optimal MS-error in the nonlinear filtering problem (1) is upper bounded by the optimal MS-error in the following linear one:

$$\begin{aligned} dz_t &= \bar{\alpha}(t)z_t dt + \sigma(t)dw_t \\ dy_t' &= \underline{\beta}(t)z_t dt + \rho(t)dv_t \end{aligned}$$

[4] The Appendix follows Section 2.

Proof:

The conditional mean $\hat{x}_t := E(x_t|\mathcal{Y}_0^t)$ and the conditional optimal MS-error

$$\hat{p}_t := E[(x_t - \hat{x}_t)^2|\mathcal{Y}_0^t]$$

are given by [16]:

$$d\hat{x}_t = \hat{g}(t, x_t)dt + \frac{\hat{e}_t}{\rho^2(t)}d\bar{w}_t; \quad \hat{x}_0 = 0$$

$$d\hat{p}_t = [\sigma^2(t) + 2((\widehat{x_t g_t}) - \hat{x}_t\hat{g}_t) - \frac{1}{\rho^2(t)}(\hat{e}_t)^2]dt + \frac{T_t}{\rho^2(t)}d\bar{w}_t \tag{11}$$

$$\hat{p}_0 = \sigma_0^2$$

where $\widehat{(\cdot)}$ denotes conditional expectation and

$$g_t = g(t, x_t); h_t = h(t, x_t)$$

$$\hat{e}_t = (x_t h_t)hat - \hat{x}_t\hat{h}_t$$

$$T_t = (\widehat{x_t^2 h_t}) - \hat{x}_t^2\hat{h}_t - 2\hat{x}_t(\widehat{x_t h_t}) + 2(\hat{x}_t)^2\hat{h}_t$$

and $d\bar{w}_t := dy_t - \hat{h}(t, x_t)dt$ is the innovation process which is a Wiener process on $\mathcal{Y}_0^t$.

Since the expectation of Ito integrals is zero and $E\hat{p}_t = E(x_t - \hat{x}_t)^2 = p(t)$, by taking the expectation on both sides of (11) we get

$$\dot{p}(t) = \sigma^2(t) + 2E((\widehat{x_t g_t}) - \hat{x}_t\hat{g}_t) - \frac{E(\hat{e}_t)^2}{\rho^2(t)}, \quad p(0) = \sigma_0^2$$

The smoothing property of conditional expectations [1] implies

$$E((\widehat{x_t g_t}) - \hat{x}_t\hat{g}_t) = E(x_t - \hat{x}_t)(g_t - g(t, \hat{x}_t))$$

$$= E\tilde{x}_t(g_t - g(t, \hat{x}_t))$$

Therefore,

$$\dot{p}(t) = \sigma^2(t) + 2E\tilde{x}_t(g_t - g(t, \hat{x}_t)) - \frac{E(\hat{e}_t)^2}{\rho^2(t)}, \quad p(0) = \sigma_0^2 \tag{12}$$

Jensen's inequality [1] implies that:

$$E(\hat{e}_t)^2 \geq (E\hat{e}_t)^2$$

$$E\hat{e}_t = E((\widehat{x_t h_t}) - \hat{x}_t\hat{h}_t) = E\tilde{x}_t(h_t - h(t,\hat{x}_t))$$

now

$$h(t,x_t) - h(t,\hat{x}_t) = \tilde{x}_t \int_0^1 h_x[t,\hat{x}_t + s\tilde{x}_t]ds := \tilde{x}_t\psi_h$$

Hence,

$$E\hat{e}_t = E\tilde{x}_t^2\psi_h$$

$\mathcal{H}_2$ implies that $\psi_h \geq \underline{\beta}(t)$ a.s.

$$E\hat{e}_t \geq \underline{\beta}(t)E\tilde{x}_t^2 = \underline{\beta}(t)p(t) \tag{13}$$

$$E(\hat{e}_t)^2 \geq (E\hat{e}_t)^2 \geq \underline{\beta}^2(t)p^2(t)$$

Similarly $\mathcal{H}_1$ implies that

$$E\tilde{x}_t(g_t - g(t,\hat{x}_t)) = E\psi_g\tilde{x}_t^2 \leq \bar{\alpha}(t)E\tilde{x}_t^2 = \bar{\alpha}(t)p(t) \tag{14}$$

Combining (12)-(14) and using the comparison theorem in the appendix yields: $p(t) \leq u(t)$.

QED

An alternate and more constructive approach to getting the same result, due to A.S. Gilman and I.B. Rhodes [13], is outlined below. The upper bound is derived by considering the following family of parametrized nonlinear suboptimal filters, the structure of which is suggested by the Kalman filter:

$$dx_t^{(k)} = g(t,x_t^{(k)})dt + k(t)[dy_t - h(t,x_t^{(k)})dt], \quad x_0^{(k)} = 0$$

where $k(t)$ is a non random, continuous non-negative bounded function.

To each gain $k(t)$ is associated a suboptimal filter given by (15) and denoted $\{x\}_k$. It can be shown ([13,20]) that:

1. Corresponding to each $\{x\}_k$ there exists a function $u_k(t)$ satisfying the linear ODE:

$$\dot{u}_k(t) = \sigma^2(t) + \rho^2(t)k^2(t) + 2[\bar{\alpha}(t) - k(t)\underline{\beta}(t)]u_k(t); \quad u_k(0) = \sigma_0^2 \tag{16}$$

such that

$$p^k(t) := E(x_t - x_t^k)^2 \leq u_k(t) \tag{17}$$

2. The suboptimal filter $\{x\}_{k^*}$ obtained for the particular choice $k^*(t) = \frac{\beta(t)}{\rho^2(t)}u(t)$ i.e.,

$$dx_t^* = g(t, x_t^*)dt + \frac{\beta(t)}{\rho^2(t)}u(t)[dy_t - h(t, x_t^*)dt]; \quad x_0^* = 0 \tag{18}$$

where $u(t)$ satisfies the Riccati equation:

$$\dot{u}(t) = \alpha^2(t) + 2\bar{\alpha}(t)u(t) - \frac{\beta^2(t)}{\rho^2(t)}u^2(t); \quad u(0) = \sigma_0^2 \tag{19}$$

is such that $u(t) \leq u^k(t)$ for every continuous nonnegative function $k(t)$. More importantly, we have the following inequalities:

$$p(t) := E(x_t - E(x_t|\mathcal{Y}_0^t))^2 \leq p^*(t) := E(x_t - x_t^*)^2 \leq u(t) \tag{20}$$

The nonlinear filter given by *(18)-(19), subsequently refered to as the* bound optimal filter (BOF), will turn out to be near optimal in many situations of practical importance as will be seen in the next subsection and in [21].

1.2.3 Summary

For systems modeled by one dimensional Ito SDE's of the form:

$$\begin{aligned} dx_t &= g(t, x_t)dt + \sigma(t)dw_t \quad &(21)\\ dy_t &= h(t, x_t)dt + \rho(t)dv_t \quad &(22) \end{aligned}$$

$$Ex_0 = 0, Ex_0^2 = \sigma_0^2$$

with g and h satisfying

$$\mathcal{H}_1 : |g_x(t, x) - \alpha(t)| \prec \Delta\alpha(t) \text{ denoted by } g \in \prec [\alpha(t), \Delta\alpha(t)] \tag{23}$$

$$\mathcal{H}_2 : |h_x(t, x) - \beta(t)| \prec \Delta\beta(t) \text{ denoted by } h \in \prec [\beta(t), \Delta\beta(t)] \tag{24}$$

define

$$\bar{\alpha}(t) := \alpha(t) + \Delta\alpha(t); \quad \underline{\alpha}(t) := \alpha(t) - \Delta\alpha(t) \tag{25}$$

$$\bar{\beta}(t) := \beta(t) + \Delta\beta(t); \quad \underline{\beta}(t) := \beta(t) - \Delta\beta(t) \geq 0 \tag{26}$$

$$p(t) := E(x_t - E(x_t|\mathcal{Y}_0^t))^2 \tag{27}$$

$$p^*(t) := E(x_t - x_t^*)^2 \tag{28}$$

where x_t^* is the BOF and is given by

$$dx_t^* = g(t, x_t^*)dt + \frac{\underline{\beta}^2(t)}{\rho^2(t)}u(t)[dy_t - h(t, x_t^*)dt]; \quad x_0^* = 0 \tag{29}$$

$$\dot{u}(t) = \sigma^2(t) + 2\bar{\alpha}(t)u(t) - \frac{\underline{\beta}^2(t)}{\rho^2(t)}u^2(t); \quad u(0) = \sigma_0^2. \tag{30}$$

Then by combining the results from the previous two sections we readily get the following bounds on the optimal MS-error:

$$0 \leq \ell(t) \leq p(t) \leq p^*(t) \leq u(t) \tag{31}$$

where

$$\dot{\ell}(t) = \sigma^2(t) + 2\underline{\alpha}(t)\ell(t) - \frac{1}{\rho^2(t)}[\bar{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}(\Delta\alpha(t))^2]\ell^2(t) \tag{32}$$

$$\ell(0) = \sigma_0^2$$

and $u(t)$ satisfies (30).

1.3 Asymptotically Linear Systems

In this section we discuss systems that are asymptotically time invariant, i.e.,

$$\begin{aligned} dx_t &= g(t, x_t)dt + \sigma dw_t & (33) \\ dy_t &= h(t, x_t)dt + \rho dv_t & (34) \end{aligned}$$

where

$$\begin{aligned} g(t, x) &= g(x) + \lambda(t)f(t, x) \\ h(t, x) &= h(x) + \nu(t)k(t, x) \end{aligned}$$

$$g \in\prec [a, \Delta a] \quad ; \quad f \in\prec [\mu(t), \Delta\mu(t)]$$
$$h \in\prec [c, \Delta c] \quad ; \quad k \in\prec [\varsigma(t), \Delta\varsigma(t)]$$

and

$$\lim_{t\to\infty}[\lambda(t), \nu(t)] = [0, 0] \tag{35}$$

In the particular case where $g(x)$ and $h(x)$ are linear (the limiting system is linear), one is interested in knowing whether the Kalman filter (KF) designed formally for the limiting linear system and driven by (the nonlinear observations) y_t in (33) is asymptotically optimal as t becomes large. This situation arises for example when the nonlinearities are neglected during the modelization process. Using an abuse of terminology, the nonlinear filter resulting from the scheme just described will be (wrongly) called to as the "KF."

More specifically, let

$$dx_t = ax_t dt + \lambda(t) f(t, x_t)dt + \sigma dw_t \tag{36}$$
$$dy_t = cx_t dt + \nu(t)k(t, x_t)dt + \rho dv_t \tag{37}$$

$$Ex_0 = 0, Ex_0^2 = \sigma_0^2 > 0$$

Then the "KF" designed for the limiting system is

$$dx_t^k = ax_t^k dt + \frac{c}{\rho^2}r(t)[dy_t - cx_t^k dt]; x^k(0) = 0 \tag{38}$$

$$\dot{r}(t) = \sigma^2 + 2ar - \frac{c^2}{\rho^2}r^2; r(0) = \sigma_0^2 \tag{39}$$

and the questions of interest are:

- Under what conditions is x_t^k (or the BOF x_t^*) asymptotically optimal as $t \to \infty$, i.e. $\lim_{t\to\infty}(p^k(t) - p(t)) = 0$ ($\lim_{t\to\infty}(p^*(t) - p(t)) = 0$)? where

$$p^k(t) = E(x_t - x_t^k)^2 \tag{40}$$
$$p^*(t) = E(x_t - x_t^*)^2 \tag{41}$$
$$p(t) = E(x_t - E(x_t|\mathcal{Y}_0^t))^2 \tag{42}$$

- Would the same result hold for the steady state "KF" ("SSKF"), obtained by setting $r(t) = r(\infty)$ in (38)?

The bounds on the optimal MS-error derived in the previous section are used to answer these questions in the linear limiting case. However, the bounds on the partial derivatives of the nonlinearities do not contain "enough information" to treat similar questions in the general case where $g(x)$ and $h(x)$ are nonlinear.

Consequently, we will only consider the class of nonlinear filtering problems (36) with the assumptions:

$\mathcal{H}_1:\ f \in\prec [\mu(t), \Delta\mu(t)]; \quad k \in\prec [\varsigma(t), \Delta\varsigma(t)]$

$\mathcal{H}_2:\ \lambda(t)$ and $\nu(t)$ are continuous, vanishing functions on $[0,\infty[$ and non-negative for simplicity

$\mathcal{H}_3:\ \mu(t), \Delta\mu(t), \varsigma(t)$ and $\Delta\varsigma(t)$ are bounded continuous functions on $[0,\infty[$.

$\mathcal{H}_4:\ c + \nu(t)\underline{\varsigma}(t) \geq \delta_0 > 0; \quad c \neq 0.$

In the next two subsections we show that:

$$\lim_{t\to\infty}(p^*(t) - p(t)) = 0 \text{ and } \lim_{t\to\infty}(p^k(t) - p(t)) = 0 \tag{43}$$

this is done by bounding $p(t)$ as

$$0 \leq \ell(t) \leq p(t) \leq p^*(t) \leq u(t) \tag{44}$$

$$0 \leq \ell(t) \leq p(t) \leq p^k(t) \leq q(t) \tag{45}$$

and showing that

$$\lim_{t\to\infty}(u(t) - \ell(t)) = 0 \text{ and } \lim_{t\to\infty}(q(t) - \ell(t)) = 0 \tag{46}$$

The result is then generalized to the case

$$g(t,x) = ax + \sum_{i=1}^{n} \lambda_i(t) f_i(t,x) \tag{47}$$

$$h(t,x) = cx + \sum_{i=1}^{m} \nu_i(t) k_i(t,x) \tag{48}$$

which in turn can be applied to treat cases where a and c are time varying functions.

1.3.1 Asymptotic optimality of the BOF

In the case of (36), we note that $\mathcal{H}_1$ and $\mathcal{H}_2$ imply:

$$g(t,x) = ax + \lambda(t)f(t,x) \in\prec [a + \lambda(t)\mu(t), \lambda(t)\Delta\mu(t)] \tag{49}$$

$$h(t,x) = cx + \nu(t)k(t,x) \in\prec [c + \nu(t)\varsigma(t), \nu(t)\Delta\varsigma(t)] \tag{50}$$

Thus the results in section 1.2.3 apply with

$$\bar{\alpha} = a + \lambda(t)\mu(t) + \lambda(t)\Delta\mu(t) = a + \lambda(t)\bar{\mu}(t) \tag{51}$$

$$\underline{\alpha} = a + \lambda(t)\underline{\mu}(t) \tag{52}$$

$$\bar{\beta} = c + \nu(t)\bar{\varsigma}(t) \tag{53}$$

$$\underline{\beta} = c + \nu(t)\underline{\varsigma}(t) \tag{54}$$

and the BOF is given here by:

$$dx_t^* = ax_t^* dt + \lambda(t)f(t,x_t^*)dt + \frac{\beta(t)}{\rho^2}u(t)[dy_t - cx_t^* dt - \nu(t)k(t,x_t^*)dt] \tag{55}$$

$$x_0^* = 0$$

$$\dot{u}(t) = \sigma^2 + 2\bar{\alpha}u(t) - \frac{\beta^2}{\rho^2}u^2(t); \qquad u(0) = \sigma_0^2 \tag{56}$$

The asymptotic optimality of the BOF is a direct consequence of the following Lemma:

Lemma 3-1: *Let $\theta_1, \theta_2, \gamma_1$ and γ_2 be continuous functions on $[0, +\infty[$ such that*

$$\lim_{t\to\infty} \theta_i(t) = a$$

$$\lim_{t\to\infty} \gamma_i^2(t) = c^2; t \geq 0; i = 1, 2$$

and consider the Riccati equations:

$$\dot{v}_1 = \sigma^2 + 2\theta_1 v_1 - \frac{\gamma_1^2}{\rho^2}v_1^2; \qquad v_1(0) = \sigma_0^2 \tag{57}$$

$$\dot{v}_2 = \sigma^2 + 2\theta_2 v_2 - \frac{\gamma_2^2}{\rho^2}v_2^2; \qquad v_2(0) = \sigma_0^2 \tag{58}$$

If $v_1(t) \geq v_2(t)$ and if one of the assumptions given below holds then:

$$\lim_{t\to\infty} v_1(t) = \lim_{t\to\infty} v_2(t)$$

Assumptions:

$$A_1 : a < 0$$

$$A_2 : v_2(t) \geq r(t), t \geq 0 \text{ and } \gamma_1^2 \geq \delta^2 > 0 \text{ for some } \delta$$

Recall that:

$$\dot{r}(t) = \sigma^2 + 2ar - \frac{c^2}{\rho^2} r^2, \quad r(0) = \sigma_0^2 \tag{59}$$

Proof:

Let $w(t) = v_1(t) - v_2(t) \geq 0$. Then a straightforward computation yields

$$\dot{w} = 2(\theta_1 - \theta_2)v_2 + \frac{1}{\rho^2}(\gamma_2^2 - \gamma_1^2)v_2^2 + 2(\theta_1 - \frac{\gamma_1^2}{\rho^2}v_2)w - \frac{\gamma_1^2}{\rho^2}w^2 \tag{60}$$

$$w(0) = 0$$

which we rewrite as

$$\dot{w}(t) = i(t) + 2j(t)w - \frac{\gamma_1^2}{\rho^2}w^2, \quad w(0) = 0 \tag{61}$$

where

$$i(t) = 2(\theta_1 - \theta_2)v_2 + \frac{1}{\rho^2}(\gamma_2^2 - \gamma_1^2)v_2^2 \tag{62}$$

$$j(t) = \theta_1 - \frac{\gamma_1^2}{\rho^2}v_2 \tag{63}$$

Equation (61) clearly implies:

$$\dot{w} \leq i(t) + 2j(t)w. \tag{64}$$

Depending on the assumption used (A_1 or A_2) we will bound $w(t)$ differently using the comparison theorem.

Assumption A_1:

Since $\ell(t)$ and $w(t)$ are nonnegative, $w(t)$ can be bounded as

$$\dot{w} \leq i(t) + 2\theta_1 w \tag{65}$$

thus $0 \leq w(t) \leq z(t)$ where

$$\dot{z}(t) = i(t) + 2\theta_1 z; \qquad z(0) = 0 \tag{66}$$

Similarly $v_1(t) \leq V_1(t)$ where

$$\dot{V}_1 = \sigma^2 + 2\theta_1 V_1, \qquad V_1(0) = \sigma_0^2 \tag{67}$$

If $a < 0$ then $\lim_{t\to\infty} \theta_1 = a < 0$ and Perron's theorem (see the appendix) can be applied to (66) and (67). We get

$$V_1(\infty) = -\frac{\sigma^2}{2a}. \tag{68}$$

Since $v_2(t) \leq v_1(t) \leq V_1(t)$ for every $t \geq 0$, (68) implies

$$\lim_{t\to\infty} i(t) = 0$$

Re–applied to (66) Perron's theorem yields

$$\lim_{t\to\infty} z(t) = 0 \text{ that is } \lim_{t\to\infty} w(t) = 0$$

Assumption A_2:

Since $v_2(t) \geq r(t), j(t) \leq \theta_1 - \frac{\gamma_1^2}{\rho^2} r(t)$, (64) then implies that $w(t) \leq z(t)$, where:

$$\dot{z} = i(t) + 2(\theta_1 - \frac{\gamma_1^2}{\rho^2} r(t))z(t); \qquad z(0) = 0 \tag{69}$$

$\lim_{t\to\infty}(\theta_1 - \frac{\gamma_1^2}{\rho^2} r(t)) = a - \frac{c^2}{\rho^2} r(\infty)$; but $r(\infty)$ is the positive root of

$$\sigma^2 + 2ax - \frac{c^2}{\rho^2} x^2 = 0 \tag{70}$$

i.e.,

$$r(\infty) = \frac{\rho^2}{c^2}[a + (a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2}]$$

and

$$a - \frac{c^2}{\rho^2} r(\infty) = -(a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2}.$$

Thus $\lim_{t\to\infty} z(t) = 0$ provided $\lim_{t\to\infty} \dot{z}(t) = 0$. For this to happen it suffices that $v_2(t)$ be bounded ($v_1(t)$ be bounded). Using the assumptions and the comparison theorem, we immediately get $v_1(t) \leq V_1(t)$ where

$$\dot{V}_1 = \sigma^2 + 2\theta_M V_1 - \frac{\delta^2}{\rho^2} V_1^2; \quad V_1(0) = \sigma_0^2 \tag{71}$$

and θ_M is a nonzero upper bound of $\theta_1(t)$. $V_1(t)$ is clearly bounded. We conclude that $\lim_{t\to\infty} z(t) = 0$, i.e., $\lim_{t\to\infty} w(t) = 0$.

Note: We can conclude that $v_1(\infty) = v_2(\infty) = r(\infty)$ provided one of the following holds:

1. $v_1(t) \geq v_2(t) \geq r(t)$; $t \geq 0$
2. $v_1(t) \geq r(t) \geq v_2(t)$; and $a < 0$
3. $r(t) \geq v_1(t) \geq v_2(t)$ and $a < 0$.

This last assertion is obtained by applying the above Lemma to the pair (r, v_2).

Proposition 3-2: *If $\mathcal{H}_1 - \mathcal{H}_4$ and $\mathcal{H}_5$ or $\mathcal{H}_6$ hold, where*

$\mathcal{H}_5$: $a < 0$

$\mathcal{H}_6$: $\ell(t) \geq r(t), \quad t \geq 0$

then the BOF given by (55)-(56) is asymptotically optimal as $t \to \infty$.

Proof:

We have that: $0 \leq \ell(t) \leq p(t) \leq p^*(t) \leq u(t)$ where $\ell(t)$ and $u(t)$ are given by (51)(52)(56) and (58) in section 1.2.3. Lemma 3-1 can then be applied to $u(t)$ and $\ell(t)$ by taking:

$$\theta_1(t) = \bar{\alpha}(t) = a + \lambda(t)\bar{\mu}(t) \tag{72}$$

$$\theta_2(t) = \underline{\alpha}(t) = a + \lambda(t)\underline{\mu}(t) \tag{73}$$

$$\gamma_1^2(t) = \underline{\beta}^2(t) = [c + \nu(t)\underline{\varsigma}(t)]^2 \tag{74}$$

$$\gamma_2^2(t) = \bar{\beta}^2(t) + 4\frac{\rho^2}{\sigma^2}(\Delta\alpha(t))^2$$

$$\gamma_2^2 = [c + \nu(t)\bar{\varsigma}(t)]^2 + 4\frac{\rho^2}{\sigma^2}\lambda^2(t)(\Delta\mu(t))^2 \tag{75}$$

It is readily checked that all hypotheses in Lemma 3-1 are satisfied and the result follows.

Remarks:

(1) It follows directly from Lemma 3-1 that if $\mathcal{H}_5$ is replaced by $\mathcal{H}_5'$: $a < 0$ and either $u(t) \geq r(t)$ or $u(t) \leq r(t)$ then $p(\infty) = p^*(\infty) = r(\infty) = \frac{\rho^2}{c^2}[a + (a^2 + \frac{\sigma^2}{\rho^2}c^2)^{1/2}]$

(2) A sufficient condition for $\mathcal{H}_6$ to hold is $\underline{\mu} \geq 0$ and $(1 + \nu\bar{\varsigma}/c)^2 + 4\lambda^2\frac{\rho^2}{\sigma^2}\frac{(\Delta\mu)^2}{c^2} < 1$ for every $t \geq 0$.

Assuming that $c > 0$ and rewriting the last inequality as

$$2\nu\frac{\bar{\varsigma}}{c} + \nu^2\frac{\bar{\varsigma}^2}{c^2} + 4\lambda^2\frac{\rho^2}{\sigma^2}\frac{(\Delta\mu)^2}{c^2} < 0,$$

it can be seen that a necessary condition for this last inequality to hold is $\bar{\varsigma} \leq 0$. It turns out that $\mathcal{H}_6$ holds in many cases if f and k lie in the first/third quadrant ($\underline{\mu} \geq 0$) and second/fourth quadrant ($\bar{\varsigma} \leq 0$) respectively (e.g. see Example (2) below).

In general hypotheses such as $\mathcal{H}_6$ should be checked numerically.

Next we generalize Proposition 3-2 to nonlinearities of the following type:

$$g(t,x) = ax + \sum_{i=1}^{n} \lambda_i(t) f_i(t,x) \tag{76}$$

$$h(t,x) = cx + \sum_{j=1}^{m} \nu_j(t) k_j(t,x) \tag{77}$$

with the assumptions $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ holding for each $i = 1,\ldots,n; j = 1,\ldots,m$.

Using a vector notation, e.g., $\Delta\mu = (\Delta\mu_1, \ldots, \Delta\mu_n)^T$, and $< \cdot, \cdot >_n$ to denote the inner product in R^n, the nonlinearities above can be written in the more condensed form:

$$g(t,x) = ax + < \lambda(t), f(t,x) >_n \tag{78}$$

$$h(t,x) = cx + < \nu(t), k(t,x) >_m \tag{79}$$

and we clearly have

$$g \in \prec [a + < \lambda, \mu >_n ; < \lambda, \Delta\mu >_n] \tag{80}$$

$$h \in \prec [c + < \nu, \varsigma >_m ; < \nu, \Delta\varsigma >_m] \tag{81}$$

Thus, if we make the additional hypothesis $\mathcal{H}_4' : \underline{\beta} = c + < \nu, \underline{\beta} >_m \geq \delta_0 > 0$ then the same results hold. More precisely the BOF is given by

$$dx_t^* = ax_t^* dt + < \lambda(t), f(t, x_t^*) > dt \tag{82}$$

$$+ \frac{\beta(t)}{\rho^2} u^2(t)[dy_t - cx_t^* dt - < \nu(t), k(t, x_t^*) > dt]; \qquad x^*(0) = 0$$

with the corresponding MSE $p^*(t)$ and the optimal MS-error $p(t)$ satisfying

$$0 \leq \ell(t) \leq p(t) \leq p^*(t) \leq u(t) \tag{83}$$

where $\ell(t)$ and $u(t)$ are given in section 1.2.3 with

$$\bar{\alpha}(t) = a + < \lambda(t), \bar{\mu}(t) > \tag{84}$$

$$\underline{\alpha}(t) = a + < \lambda(t), \underline{\mu}(t) > \tag{85}$$

$$\Delta\alpha = < \lambda(t), \Delta\mu > \tag{86}$$

$$\bar{\beta}(t) = c + < \nu(t), \bar{\varsigma}(t) > \tag{87}$$

$$\underline{\beta}(t) = c + < \nu(t), \underline{\varsigma}(t) > \tag{88}$$

The following corollary is now a direct application of Lemma 3-1.

Corollary 3-3: *If $\mathcal{H}_1 - \mathcal{H}_4'$ and $\mathcal{H}_5$ or $\mathcal{H}_6$ stated below are satisfied, then the BOF (82) is asymptotically optimal as $t \to \infty$.*

$\mathcal{H}_5 :\ a < 0$

$\mathcal{H}_6$: $\ell(t) \geq r(t); r(t)$ *given by* (59).

The corollary can be used to treat the more general cases where a and c are time varying, i.e.,

$$g(t,x) = a(t)x + \sum_{1}^{n} \lambda_i(t) f_i(t,x) \tag{89}$$

$$h(t,x) = c(t)x + \sum_{1}^{m} \nu_i(t) k_i(t,x) \tag{90}$$

where $\lim_{t\to\infty} a(t) = a$ and $\lim_{t\to\infty} c(t) = c$

As an illustration, assume that $a(t)$ and $c(t)$ are monotone and continuous, then (89) may be rewritten as

$$g(t,x) = ax + (a(t) - a)x + < \lambda, f >_n \tag{91}$$

$$h(t,x) = cx + (c(t) - c)x + < \nu, k >_m \tag{92}$$

By letting:

$$\begin{aligned} \lambda_{n+1}(t) &= |a(t) - a| \\ nu_{m+1}(t) &= |c(t) - c| \\ f_{n+1}(t,x) &= \text{sign } (a(t) - a)x \\ k_{m+1}(t,x) &= \text{sign } (c(t) - c)x \end{aligned}$$

Equation (91) becomes:

$$g(t,x) = ax + < \lambda, f >_{n+1} \tag{93}$$

$$h(t,x) = cx + < \nu, k >_{m+1} \tag{94}$$

and we are in position to apply the Corollary since λ_{n+1} and ν_{m+1} are continuous vanishing nonnegative functions with f_{n+1} and k_{m+1} belonging to $<$ [sign $(\lambda_{n+1}), \delta]$ and $<$ [sign $(\nu_{m+1}), \delta]$ respectively, where $\delta \geq 0$ is arbitrary.

1.3.2 Asymptotic optimality of the KF

For the nonlinear filtering problem (36), it is clear that (38)-(39) correspond to a regular Kalman filter designed for the underlying linear system obtained

when one ignores the nonlinear terms in (36). It should be noted however that (38)-(39) is driven by observations from a nonlinear system. We will, nevertheless, continue to refer to it as the "KF" and "SSKF" (steady state) when $r(t)$ is replaced by $r(\infty)$.

In addition to $\mathcal{H}_1$–$\mathcal{H}_4$, we make the following assumption:

$\mathcal{H}_0$: $f(t,0)$ and $k(t,0)$ are continuous, bounded on $[0,\infty[$.

Proposition 3-4: *If $a < 0$ then both the "KF" and the "SSKF" are asymptotically optimal as $t \to \infty$. Moreover:*

$$p(\infty) = p^k(\infty) = r(\infty) = \frac{\rho^2}{c^2}[a + (a^2 + \frac{\sigma^2}{\rho^2}c^2)^{1/2}] \tag{95}$$

Proof: We first derive an upper bound on $p^k(t) := E(x_t - x_t^k)^2$ where x_t^k is given by (38)-(39).

Let $\bar{x}_t = x_t - x_t^k$; then (36) and (38) yield

$$d\bar{x}_t = [\bar{g}_t - G(t)\bar{h}_t]dt + \sigma dw_t - \rho G(t) dv_t \tag{96}$$

where

$$G(t) = \frac{c}{\rho^2}r(t) \text{ (or } \frac{c}{\rho^2}r(\infty)) \tag{97}$$

$$\bar{g}_t = a\bar{x}_t + \lambda(t)f(t,x_t) \tag{98}$$

$$\bar{h}_t = c\bar{x}_t + \nu(t)k(t,x_t) \tag{99}$$

Applying Ito's chain rule gives

$$d\bar{x}_t^2 = [\sigma^2 + \rho^2 G^2(t)]dt + 2\bar{x}_t d\bar{x}_t \tag{100}$$

Taking the expectation on both sides yields:

$$\frac{d}{dt}E\bar{x}_t^2 = \dot{p}^k(t) = \sigma^2 + \rho^2 G^2(t) + 2E\bar{x}_t[\bar{g}_t - G(t)\bar{h}_t] \tag{101}$$

$$\frac{dp^k}{dt} = \sigma^2 + \rho^2 G^2 + 2E\bar{x}_t\bar{g}_t - 2GE\bar{x}_t\bar{h}_t, \quad p^k(0) = \sigma_0^2$$

$$\dot{p}^k = \sigma^2 + \rho^2 G^2 + 2(a - cG)p^k + 2\lambda E\bar{x}_t f(t, x_t) - 2\nu G E\bar{x}_t k(t, x)$$

Clearly,

$$2E\bar{x}_t f(t, x_t) \leq E\bar{x}_t^2 + Ef^2(t, x_t) = p^k(t) + Ef^2(t, x_t)$$

$$-2E\bar{x}_t k(t, x_t) \leq p^k(t) + Ek^2(t, x_t)$$

By the comparison theorem : $p^k(t) \leq q(t)$; $\quad q(0) = \sigma_0^2$ where

$$\dot{q}(t) = \sigma^2 + \rho^2 G^2 + 2(a - cG)q + \lambda(q + Ef^2) + \nu G(q + Ek^2)$$

$$= \sigma^2 + \rho^2 G^2 + \lambda E f^2 + \nu G E k^2 + [2(a - cG) + \lambda + \nu G]q$$

which we rewrite as

$$\dot{q} = i(t) + j(t)q, \quad q(0) = \sigma_0^2 \tag{102}$$

Now

$$\lim_{t\to\infty} j(t) = 2(a - \frac{c^2}{\rho^2} r(\infty)) = -2(a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2} < 0.$$

Thus, if

$$\lim_{t\to\infty} \lambda(t) E f^2(t, x_t) = \lim_{t\to\infty} \nu(t) E k^2(t, x_t) = 0 \tag{103}$$

then $\lim_{t\to\infty} i(t) = \sigma^2 + \frac{c^2}{\rho^2} r^2(\infty)$.

Applying Perron's theorem to (102) would give:

$$q(\infty) = -\frac{i(\infty)}{j(\infty)} = -\frac{\sigma^2 + \frac{c^2}{\rho^2} r^2(\infty)}{2(a - \frac{c^2}{\rho^2} r(\infty))}$$

But $r(\infty)$ satisfies the algebraic Riccati equation:

$$\sigma^2 + 2ar(\infty) - \frac{c^2}{\rho^2} r^2(\infty) = 0$$

It follows that:

$$q(\infty) = -\frac{\sigma^2 + 2ar(\infty) - \frac{c^2}{\rho^2} r^2(\infty) - 2(a - \frac{c^2}{\rho^2} r(\infty))r(\infty)}{2(a - \frac{c^2}{\rho^2} r(\infty))} = r(\infty)$$

If $a < 0$ and $\ell(t) \leq r(t)$ then by letting $v_1(t) = r(t)$ and $v_2(t) = \ell(t)$ in Lemma 3-1 we conclude that $\ell(\infty) = r(\infty) = q(\infty)$ and hence $p(\infty) = p^k(\infty) = r(\infty)$.

If $\ell(t)$ is not less or equal than $r(t)$ for every t, then we can always find a lower bound $\ell'(t)$ which is less or equal than both $\ell(t)$ and $r(t)$ (see next remark). Thus we can apply the same Lemma with $v_2(t) = \ell'(t)$ and conclude that $\ell'(\infty) = r(\infty) = q(\infty)(= \ell(\infty))$ and hence $p(\infty) = p^k(\infty) = r(\infty)$.

We now show that (103) holds if $a < 0$ and $\mathcal{H}_0$ hold. The condition $f \in\prec [\mu(t), \Delta\mu(t)]$ implies

$$\underline{\mu}(t)x + f(t,0) \leq f(t,x) \leq \bar{\mu}(t)x + f(t,0) \tag{104}$$

where the time functions $\underline{\mu}(t), \bar{\mu}(t)$ and $f(t,0)$ are all bounded continuous for $t \geq 0$. Equation (104) implies in turn that:

$$f^2(t,x) \leq A^2(t)x^2 + B^2(t) \tag{105}$$

for some continuous bounded functions A and B. Therefore,

$$\lim_{t\to\infty} \lambda(t)Ef^2(t,x_t) = 0$$

holds if

$$\lim_{t\to\infty} \lambda(t)Ex_t^2 = 0 \tag{106}$$

Ex_t^2 satisfies the following ODE [16]:

$$\frac{d}{dt}Ex_t^2 = 1 + 2\lambda(t)Ex_tf(t,x_t) + 2aEx_t^2 \tag{107}$$

$$2Ex_tf(t,x_t) \leq Ex_t^2 + Ef^2(t,x_t)$$

Using (106) and (105) in (107), we conclude by the comparison theorem that Ex_t^2 is bounded by $V(t)$ where:

$$\dot{v} = 1 + \lambda(t)B^2(t) + (2a + \lambda(t) + \lambda(t)A^2(t))V(t)$$

Perron's theorem applies and $V(\infty) = -1/2a$. Therefore,

$$\lim_{t\to\infty} \lambda(t)Ef^2(t,x_t) = 0.$$

QED

Clearly, the same thing is also true for $\nu(t)Ek^2(t,x_t)$.

Remark: Let $f \in\prec [\mu(t), \Delta\mu(t)]$ and $k \in\prec [\varsigma(t), \Delta\varsigma(t)]$ i.e.,

$$f_x \in [\underline{\mu}(t), \bar{\mu}(t)], k_x \in [\underline{\varsigma}(t), \bar{\varsigma}(t)] \tag{108}$$

The lower bound is then given by:

$$\dot{\ell}(t) = \sigma^2(t) + 2\underline{\alpha}(t)\ell(t) - \frac{1}{\rho^2}[\bar{\beta}^2 + 4\frac{\rho^2}{\sigma^2}(\Delta\alpha)^2]\ell^2(t); \quad \ell(0) = \sigma_0^2$$

where $\underline{\alpha}(t) = a + \lambda(t)\underline{\mu}(t), \Delta\alpha(t) = \lambda(t)\Delta\mu(t)$ and $\bar{\beta}(t) = c + \nu(t)\bar{\varsigma}(t)$.

Clearly, $\ell(t) \leq r(t)$ if $\underline{\mu}(t) \leq 0$ and $\bar{\varsigma}(t) \geq 0$ where

$$\dot{r}(t) = \sigma^2 + 2ar(t) - \frac{c^2}{\rho^2}r^2(t); \quad r(0) = \sigma_0^2$$

If $\underline{\mu}(t) \leq 0$ and $\bar{\varsigma}(t) \geq 0$ does not hold, then we can always choose a worse lower bound $\ell'(t)$ such that $\ell'(t) \leq r(t)$. This is possible since (108) implies that $f_x \in [\underline{\mu}'(t), \bar{\mu}(t)]; k_x \in [\underline{\varsigma}, \bar{\varsigma}'(t)]$ with $\underline{\mu}'(t) \leq 0$ and $\bar{\varsigma}'(t) \geq 0$.

Let us now turn to the case where g and h are again given by (89). Then under the same assumptions and notations of Corollary 3-3 and the additional obvious additional assumptions introduced by $\mathcal{H}_0$ (namely that it holds for each f_i, k_j), it can be shown [20] that the following holds:

Corollary 3-5: *If $a < 0$, then both the "KF" and the "SSKF" are asymptotically optimal as $t \to \infty$. Moreover,*

$$p(\infty) = p^k(\infty) = r(\infty) = \frac{c^2}{c^2}[a + (a^2\frac{\sigma^2}{\rho^2}c^2)^{1/2}] \tag{109}$$

1.4 An Example

Example:

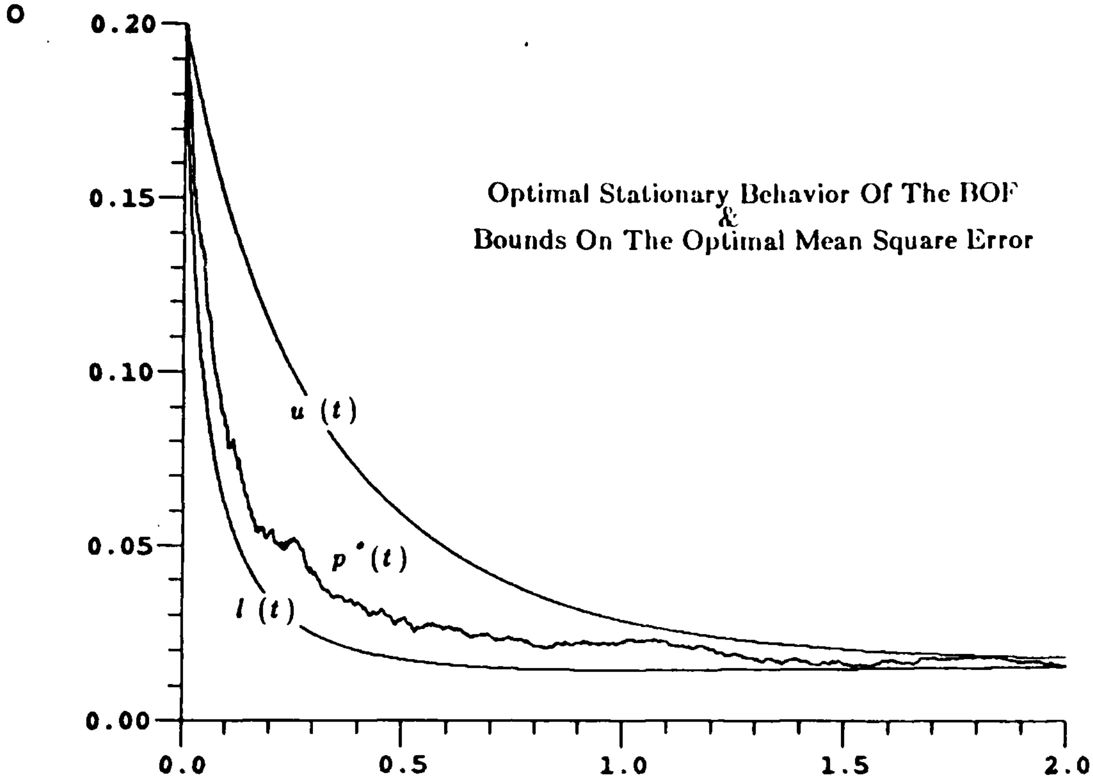

Figure 1: BOF performance

Let x_t and y_t be given by:

$$\begin{aligned} dx_t &= ax_t dt + e^{-t}\sin^2(\omega t)\tanh(x_t)dt + \sigma dw_t \\ dy_t &= cx_t dt + \frac{1}{t^2+1}x_t e^{-x_t^2}dt + \rho dv_t \end{aligned} \tag{110}$$

$$x_0 \sim \mathcal{N}(m_0, \sigma_0^2)$$

Thus,

$$\lambda(t) = e^{-t}\sin^2(\omega t); \quad f(x) = \tanh(x)\nu(t) = \frac{1}{t^2+1}; \quad k(x) = xe^{-x^2}$$

Simulations were done with the following numerical data:

$$a = -1, \omega = 50, \sigma = \rho = 0.2c = 1, m_0 = 0.0, \sigma_0^2 = 0.2 \tag{111}$$

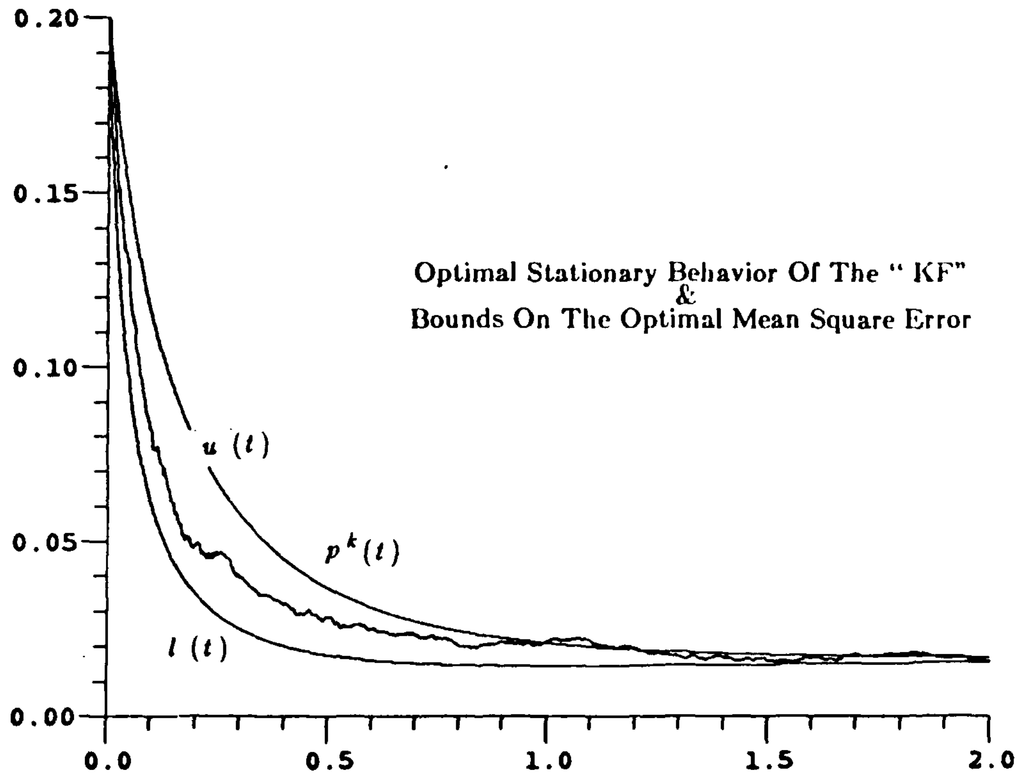

Figure 2: KF performance

for which it is readily obtained that $f \in\prec [-\frac{1}{2}, \frac{1}{2}]$ and $k \in\prec [\frac{1-2e^{-\frac{3}{2}}}{2}, \frac{1+2e^{-\frac{3}{2}}}{2}]$ i.e.,

$$\underline{\mu} = 0, \bar{\mu} = 1, \mu = \frac{1}{2}, \Delta\mu = \frac{1}{2} \tag{112}$$

$$\underline{\varsigma} = -2e^{-\frac{3}{2}}, \bar{\varsigma} = 1, \varsigma = \frac{1-2e^{-\frac{3}{2}}}{2}, \Delta\varsigma = \frac{1+2e^{-\frac{3}{2}}}{2}$$

The simulation results, obtained using Monte Carlo methods, are summarized in the plots of Figures 1 and 2 corresponding to the BOF and "KF" respectively. In Figure 1, the upper and lower bound $(u(t)\ ,\ \ell(t))$ on the optimal MS-error $p(t) := E[x_t - E(x_t|\mathcal{Y}_0^t)]^2$ together with the MSE corresponding to the BOF are plotted. A similar is given in Figure 2 for the "KF" except that instead of $u(t)$, $r(t)$ is plotted. (Recall that while u was shown to be an upper bound on the BOF MSE $p^*(t)$, neither u nor r are known to be upper bounds for the "KF" MSE $p^k(t)$).

It can be seen that the BOF and the "KF" are indeed both asymptotically optimal in the sense that $\lim_{t\to\infty}(p(t) - p^*(t)) = 0$ and $\lim_{t\to\infty}(p(t) - p^k(t)) = 0$ respectively. Moreover:

$$p(\infty) = p^*(\infty) = p^k(\infty) = r(\infty) = \frac{\rho^2}{c^2}[a + (a^2 + \frac{\sigma^2}{\rho^2}c^2)^{\frac{1}{2}}] = 0.017 \tag{113}$$

1.5 Conclusions

We investigated the asymptotic behavior question of one dimensional nonlinear filtering problems involving drifts with bounded derivatives using an upper and lower bound approach to show that the a priori mean square error associated with some suboptimal filters approaches the optimal one asymptotically. The upper and lower bounds satisfy ordinary differential equations of the Riccati type. In particular, it is shown that in the case of asymptotically time invariant systems for which the limiting system is linear, the "KF" and "SSKF" (designed for the limiting linear system) are asymptotically optimal as $t \to \infty$ (section 1.3). In other words the nonlinearity can be ignored as far as the long time behavior is concerned. This approach proved that significant information relevant to this type of filtering problems can be infered from the knowledge of the derivative bounds (i.e., f the cone in which the nonlinearities reside), and the main point is that, tractable bounds on the optimal MS-error, when available, can be used (in

addition to performance testing of suboptimal designs) as a study approach to tackle some questions arising in nonlinear filtering.

2 A Bound Approach to Filters for Weakly Nonlinear Systems

2.1 Introduction

In this section we again consider the Ito stochastic model:

$$\begin{aligned} dx_t &= g(t, x_t)dt + \sigma(t)dw_t & (114) \\ dy_t &= h(t, x_t)dt + \rho(t)dv_t & (115) \\ x(0) &= x_0, \quad 0 \le t \le T & (116) \end{aligned}$$

where g, h, α and ρ are smooth functions of their arguments, $\{v_t\}, \{w_t\}$ are independent Wiener processes, and x_0 is a random variable independent of $\{v_t\}, \{w_t\}$.

As we have shown in the previous section, the performance of suboptimal designs, however derived, may be based on lower and upper bounds on the optimal mean square error (MS-error) $p(t)$ [22]. This approach is used here to investigate the long time (asymptotic) behavior of a class of nonlinear filtering problems, namely weakly nonlinear systems [10] and systems with low measurement noise level [19]-[6]. Systems of the first type are modeled as:

$$\begin{aligned} dx_t &= a(t)x_t dt + \epsilon f(t, x_t)dt + \sigma(t)dw_t & (117) \\ dy_t &= c(t)x_t dt + \rho(t)dv_t \end{aligned}$$

while those of the second type are:

$$\begin{aligned} dx_t &= g(t, x_t)dt + \sigma(t)dw_t & (118) \\ dy_t &= h(t, x_t)dt + \epsilon dv_t \end{aligned}$$

It is well known that for filtering problems of this type there may be no finite set of equations which propagate the conditional mean.

We are interested in (one dimensional) suboptimal filters which are asymptotically optimal in the sense that the corresponding *a priori* mean square error (MSE) is identical, up to some power of ϵ, to the optimal one.

Weakly nonlinear systems have been studied in [6,5,4]. In [6], Brockett showed that in the general case, even to be optimal in the asymptotic sense, such filters must evolve in higher dimensional spaces than x_t does.

One question of particular interest is to study the effect of the weak nonlinearity on the filtering performance. In other words the question is whether the Kalman filter ("KF"), formally designed for the underlying linear system and driven by the observation $\{y_t\}$ in (117) is asymptotically optimal for small ϵ (notice that these are observations from a nonlinear system).

In section 2.3, it is shown that for a particular class of nonlinearities f (those with bounded derivatives), the "KF" and the so-called bound optimal filter (BOF, section 2.2), both of which are one dimensional filters with precomputable (nonrandom) gains, are asymptotically optimal as $\epsilon \to 0$.

Next, the low measurement noise case, first studied in [19]-[6], is treated in section 2.4 where the BOF and a constant gain version of it are shown to be asymptotically optimal, in addition, an even simpler (not involving the drift and linear) asymptotically optimal filter is obtained. Some of these results have been obtained in [19,6] by a different approach (e.g., a WKB procedure applied directly to the DMZ equation in Fisk-Statonovich form [19]), while here, basic bounds on the *a priori* optimal MS-error and perturbation methods are used. Examples with simulation results are provided in section 2.5.

2.2 Lower and Upper Bounds on the Optimal MS-Error

Let us consider the one dimensional version of (114) where x_0 is assumed to be $\mathcal{N}(0, \sigma_0^2)$; g and h are such that (114) has a unique solution [1], differentiable with continuous partial derivatives satisfying the following hypotheses:

$$Hh_1: \; |g_x(t,x) - \alpha(t)| < \Delta\alpha(t)$$

$$Hh_2: \; |h_x(t,x) - \beta(t)| < \Delta\beta(t), \quad \underline{\beta}(t) := \beta(t) - \Delta\beta(t) \geq 0$$

which we denote by

$$\begin{aligned} g &\in \prec [\alpha(t), \Delta\alpha(t)], \\ h &\in \prec [\beta(t), \Delta\beta(t)] \end{aligned} \tag{119}$$

define

$$\bar{\alpha}(t) := \alpha(t) + \Delta\alpha(t), \tag{120}$$

$$\underline{\alpha}(t) := \alpha(t) - \Delta\alpha(t)$$

$$\bar{\beta}(t) := \beta(t) + \Delta\beta(t), \qquad (121)$$
$$\underline{\beta}(t) := \beta(t) - \Delta\beta(t)$$

$$p(t) := E(x_t - E(x_t|\mathcal{Y}_0^t))^2 \qquad (122)$$

$$p^*(t) := E(x_t - x_t^*)^2 \qquad (123)$$

where x_t^* is the BOF and is given by

$$dx_t^* = g(t, x_t^*)dt + \frac{\underline{\beta}^2(t)}{\rho^2(t)}u(t)[dy_t - h(t, x_t^*)dt], \quad x_0^* = 0 \qquad (124)$$

$$\dot{u}(t) = \sigma^2(t) + 2\bar{\alpha}(t)u(t) - \frac{\underline{\beta}^2(t)}{\rho^2(t)}u^2(t), \quad u(0) = \sigma_0^2. \qquad (125)$$

The stochastic process satisfying the above nonlinear SDE is called the *bound optimal filter (BOF)*. Clearly, the BOF is readily implementable with precomputable (nonrandom) gain and it coincides with the Kalman filter when f and g are linear. Moreover, the BOF is "bound optimal" in the sense that, among all nonlinear filters given by 124 but with arbitrary non random, continuous gains $k(t)$, the choice $k^*(t) := \frac{\underline{\beta}(t)}{\rho^2}u(t)$ yields a nonlinear filter (the BOF) that has the tightest upper bound on the corresponding MS-error. Furthermore, this upper bound is precisely $u(t)$ (see [22,20,13]).

The following result, proved in [21,20], provides explicit lower and upper bounds on the (unknown) optimal MS-error $p(t)$.

Theorem 2-1: *Let $p(t), p^*(t)$ and $u(t)$ be as in (122),(123) and (125) respectively. Then:*

$$0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t)$$

where

$$\dot{\ell}(t) = \sigma^2(t) + 2\underline{\alpha}(t)\ell(t) - \frac{1}{\rho^2}[\bar{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}(\Delta\alpha(t))^2]l^2(t) \qquad (126)$$

$$\ell(0) = \sigma_0^2$$

Remark: Since $\ell(t)$ and $u(t)$ both satisfy ODE's of the Riccati type, the Theorem says that the optimal MS-error $p(t)$ in the nonlinear filtering problem is bounded by those in two corresponding Kalman filtering problems, the coefficients of which are obvious from (125) and (126).

Definition: *Let $\{x_t^s\}$ be any suboptimal filter, $p^s(t,\epsilon) := E(x_t - x_t^s)^2$ and $p(t,\epsilon) := E[x_t - E(x_t|\mathcal{Y}_0^t)]^2$. Then $\{x_t^s\}$ is said to be asymptotically optimal if $p(t,\epsilon)$ and $p^s(t,\epsilon)$ agree up to some power ($k \geq 1$) of ϵ in a nontrivial way.*

Proof of asymptotic optimality for a given suboptimal filter $\{x_t^s\}$ uses the argument that if one can bound $p(t,\epsilon)$, $p^s(t,\epsilon)$ as in

$$0 \leq \ell^s(t) \leq p(t,\epsilon) \leq p^s(t,\epsilon) \leq u^s(t,\epsilon)$$

for some tractable bounds ℓ^s and u^s, then it suffices to show that the first terms in the corresponding asymptotic expansions are identical.

2.3 Weakly Nonlinear Systems

Let x_t and y_t be given by

$$\begin{aligned} dx_t &= g(t,x_t)dt + \epsilon f(t,x_t) + \sigma(t)dw_t, \quad 0 \leq t \leq T \\ dy_t &= h(t,x_t)dt + \rho(t)dv_t \end{aligned} \tag{127}$$

where x_0 is $\mathcal{N}(0,\sigma_0^2)$, $\{w_t\},\{v_t\}$ are Brownian motions independent of x_0; f, g, and h have enough smoothness to guarantee the well posedness of (114).

In the case $\epsilon > 0$ is a small parameter, g and h are linear, we call these *weakly nonlinear systems (WNL)*. WNL systems were studied in [10] where it was shown that if, e.g., $f(t,x) = x^3$, then there does not exist a reduced order (i.e., one dimensional) filter which has the optimal asymptotic performance.

Our goal here is to exhibit one dimensional filters that are always asymptotically optimal for a restricted class of nonlinearities f, namely those with bounded derivatives.

In the next two subsections upper and lower bounds on $p(t) := E(x_t - E(x_t|\mathcal{Y}_0^t))^2$, $p^*(t) := E(x_t - x_t^*)^2$ and $p^k(t) := E(x_t - x_t^k)^2$ (x_t^*, x_t^k being the

BOF and "KF" estimators respectively) are used to establish that in the weakly nonlinear case, that is, in the case g and h are linear, both filters are asymptotically optimal in the sense that p, p^* and p^k are the same up to first order in ϵ.

2.3.1 Asymptotic optimality of the BOF

Let x_t and y_t be given by (127) and assume that:

$$g \in\prec [a(t), \Delta a(t)], \quad f \in\prec [\mu(t), \Delta\mu(t)]$$

$$h \in\prec [c(t), \Delta c(t)]$$

$$\underline{c}(t) := c(t) - \Delta c(t) > 0, t \geq 0$$

We recall that here the BOF x_t^* is given by:

$$dx_t^* = g(t, x_t^*)dt + \epsilon f(t, x_t^*)dt + \frac{\underline{c}(t)}{\rho^2(t)} u(t)[dy_t - h(t, x_t^*)dt] \tag{128}$$

$$x^*(0) = 0$$

$$\dot{u} = \sigma^2(t) + 2(\bar{a}(t) + \epsilon\bar{\mu}(t))u(t) - \frac{\underline{c}^2(t)}{\rho^2(t)} u^2,$$

$$u(0) = \sigma_0^2$$

Proposition 3-1: *If $\Delta a(t) = \Delta c(t) = 0$ and $c(t) > 0$, then the BOF is asymptotically optimal as $\epsilon \to 0$, i.e.,*

$$p^*(t) \sim p(t) = r(t) + O(\epsilon), 0 \leq t \leq T \tag{129}$$

where

$$\dot{r} = \sigma^2(t) + 2a(t)r(t) - \frac{c^2(t)}{\rho^2(t)} r^2, \quad r(0) = r_0^2 \tag{130}$$

Remark: If furthermore, the system is time invariant then

$$p^*(t) = p(t) = r(t) + 2\epsilon\mu \int_0^t \phi(t, s) r(s) ds + O(\epsilon, \Delta\mu) \tag{131}$$

where

$$r(t) = \frac{\rho^2}{c^2}\left\{a + \delta\frac{1 - Ae^{-2\delta t}}{1 + Ae^{-2\delta t}}\right\} \tag{132}$$

$$\delta = \sqrt{a^2 + \frac{\sigma^2}{\rho^2}c^2}$$

$$A = \frac{\frac{c^2}{\rho^2}(a+\delta) - \sigma_0^2}{\sigma_0^2 - \frac{c^2}{\rho^2}(a-\delta)}$$

$$\phi(t,s) = e^{2a(t-s)}\exp\left\{-2\frac{c^2}{\rho^2}\int_s^t r(\tau)d\tau\right\} \tag{134}$$

here $O(x,y)$ means order of each one of the arguments separately.

Proof: It readily follows from the above assumptions that $(g+\epsilon f) \in\prec [a(t)+\epsilon\mu(t), \Delta a(t)+\epsilon\Delta\mu(t)]$.

From Theorem 2-1 we get

$$0 \le \ell(t) \le p(t) \le p^*(t) \le u(t) \tag{135}$$

where:

$$\dot{u} = \sigma^2(t) + 2(\bar{a}(t) + \epsilon\bar{\mu}(t))u - \frac{\underline{c}^2(t)}{\rho^2(t)}u^2 \tag{136}$$

$$u(0) = \sigma_0^2$$

$$\dot{\ell} = \sigma^2(t) + 2(\underline{a}(t) + \epsilon\underline{\mu}(t))\ell - \frac{1}{\rho^2(t)}[\bar{c}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}(\delta a(t) + \epsilon\delta\mu(t))^2]\ell^2 \tag{137}$$

$$\ell(0) = \sigma_0^2$$

expanding $u(t)$ in the form:

$$u(t) \sim \sum_{i=0}^{inf} u_i(t)\epsilon^i \tag{138}$$

gives:

$$u^2(t) \sim \sum_{k=0}^{\infty} c_k\epsilon^k \tag{139}$$

$$c_k = \sum_{j=0}^{n} u_j(t)u_{n-j}(t)$$

Plugging (138) and (139 in (136) and equating powers of ϵ yields:

$$\dot{u}_0 = \sigma^2(t) + 2\bar{a}(t)u_0 - \frac{\underline{c}^2(t)}{\rho^2(t)}u_0^2, \quad u_0(0) = \sigma_0^2 \tag{140}$$

$$\dot{u}_1 = 2[\bar{a}(t) - \frac{\underline{c}^2(t)}{\rho^2(t)}u_0(t)]u_1 + 2\bar{\mu}(t)u_0(t), \quad u_1(0) = 0 \tag{141}$$

Proceeding similarly for $\ell(t)$, one obtains:

$$\dot{\ell}_0 = \sigma^2(t) + 2\underline{a}(t)\ell_0 - \frac{1}{\rho^2}[\bar{c}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}\delta a^2(t)]\ell_0^2$$

$$\ell_0(0) = \sigma_0^2$$

$$\dot{\ell}_1 = 2[\underline{a}(t) - \frac{1}{\rho^2(t)}(\bar{c}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}\delta a^2(t))\ell_0]\ell_1 + 2\underline{\mu}(t)\ell_0 - 8\frac{\delta a\delta\mu}{\sigma^2}\ell_0^2 \tag{143}$$

$$\ell_1(0) = 0$$

(here $\delta a^2 := (\delta a)^2$)

It is clear from (140 and (142) that $u_0(t)$ and $\ell_0(t)$ are different in the general case but coincide with $r(t)$ if $\delta a = \delta c = 0$ that is:

$$g(t,x) = a(t)x \text{ and } h(t,x) = c(t)x$$

Now if the system is time invariant i.e.,

$$a(t) = a, \quad \mu(t) = \mu, \quad c(t) = c, \quad \sigma(t) = \sigma \text{ and } \rho(t) = \rho$$

then one easily gets the results in the remark above by using the Riccati transformation $r = \frac{\rho^2}{c^2}\frac{\dot{w}}{w}$ to solve (130) and the variation of constants formula in (141) and (143).

2.3.2 Asymptotic optimality of the KF

The question considered here is whether one could, in the case of weakly nonlinear systems, ignore the nonlinear part in the drift, use the Kalman filter designed for the underlying linear system (driven by y_t) and be able to

achieve asymptotic optimality as $\epsilon \to 0$. It is important however to notice that even though this scheme is being refered to as the "KF", it has little to do with the regular Kalman filter, the reason being that the "KF" is driven by observations from a nonlinear system.

Accordingly, Let $g(t,x) = a(t)x, h(t,x) = c(t)x$ and assume that $f \in \prec [\mu(t), \delta\mu(t)], c(t) > 0$, then the "KF" is given by:

$$dx_t^k = a(t)x_t^k dt + \frac{c(t)}{\rho^2(t)} r(t)[dy_t - c(t)x_t^k dt], \quad x^k(0) = 0 \tag{144}$$

where $r(t)$ is as in (130).

Proposition 3-2: *Under the above assumption, the "KF" is asymptotically optimal as $\epsilon \to 0$ in the sense that:*

$$p^k(t) \sim p(t) = r(t) + O(\epsilon) \quad 0 \le t \le T$$

Proof: We first derive an upper bound on $p^k(t) := E(x_t - x_t^k)^2$ where x_t^k is given by (144).

Let $\tilde{x}_t := x_t - x_t^k$; then

$$d\tilde{x}_t = [\tilde{g}_t - c(t)G(t)\tilde{x}_t]dt + \sigma(t)dw_t - \rho(t)G(t)dv_t \tag{145}$$

where $G(t) := \frac{c(t)}{\rho^2} r(t)$ and $\tilde{g}_t = a(t)\tilde{x}_t + \epsilon f(t,x_t)$. Applying Ito's chain rule [16] gives

$$d\tilde{x}_t^2 = [\sigma^2 + \rho^2 G^2]dt + 2\tilde{x}_t d\tilde{x}_t \tag{146}$$

Taking expectations on both sides yields:

$$\frac{d}{dt} E\tilde{x}_t^2 = \dot{p}^k(t) = \sigma^2 + \rho^2 G^2 + 2E\tilde{x}_t[\tilde{g}_t - cG\tilde{x}_t]$$

$$\dot{p}^k(t) = \sigma^2 + \rho^2 G^2 + 2E\tilde{x}_t\tilde{g}_t - 2cG\tilde{x}_t^2, \quad p^k(0) = \sigma_0^2$$

$$\dot{p}^k = \sigma^2 + \rho^2 G^2 + 2(a - cG)p^k + 2\epsilon E\tilde{x}_t f(t,x_t) \tag{147}$$

Clearly

$$2E\tilde{x}_t f(t,x_t) \le E\tilde{x}_t^2 + Ef^2(t,x_t) = p^k(t) + Ef^2(t,x_t) \tag{148}$$

By the comparison theorem (see appendix): $p^k(t) \le q(t), q(0) = \sigma_0^2$ where

$$\dot{q}(t) = \sigma^2 + \rho^2 G^2 + 2(a - cG)q + \epsilon(q + Ef^2)$$

$$= \sigma^2 + \rho^2 G^2 + \epsilon E f^2 + [2(a - cG) + \epsilon]q \tag{149}$$

which we rewrite as

$$\dot{q} = i(t) + j(t)q, \quad q(0) = \sigma_0^2 \tag{150}$$

$$i(t) = \sigma^2 + \frac{c^2}{\rho^2} r^2 + \epsilon E f^2(t, x_t)$$

$$j(t) = \epsilon + 2[a - \frac{c^2}{\rho^2} r(t)]$$

We therefore have the following bounds:

$$\ell(t) \leq p(t) \leq p^k(t) \leq q(t) \tag{151}$$

where:

$$\dot{\ell} = \sigma^2 + 2(a + \epsilon\underline{\mu})\ell - \frac{1}{\rho^2}[c^2 + 4\frac{\rho^2}{\sigma^2}\delta\mu^2\epsilon^2]\ell^2 \tag{152}$$

$$\ell(0) = \sigma_0^2$$

Expanding $q(t)$ in the form:

$$q(t) \sim \sum_{i=0}^{\infty} q_i(t)\epsilon^i$$

and equating powers of ϵ yields:

$$\dot{q}_0 = \sigma^2(t) + \frac{c^2(t)}{\rho^2(t)} r^2(t) + 2[a(t) - \frac{c^2(t)}{\rho^2(t)} r(t)]q_0, \quad q_0(0) = \sigma_0^2$$

Let $w := q_0(t) - \ell_0(t)$. Then from the previous section it follows by making *$\delta a = 0$ in (142) that $w(t) = q_0(t) - r(t)$. By differentiating we get*

$$\dot{w}(t) = \frac{c^2(t)}{\rho^2(t)} r^2(t) + 2[a(t) - \frac{c^2(t)}{\rho^2(t)} r(t)]q_0^\epsilon - 2a(t)r(t) + \frac{c^2(t)}{\rho^2(t)} r^2(t)$$

This in turn easily becomes:

$$\dot{w} = 2[a(t) - \frac{c^2(t)}{\rho^2(t)} r(t)]w, \quad w(0) = 0$$

The solution of which clearly is $w(t) = 0$ which implies $q_0 = r$.

2.4 Low Measurement Noise Level

Consider the system:

$$\begin{aligned} dx_t &= g(t, x_t)dt + \sigma(t)dw_t \\ dy_t &= h(t, x_t)dt + \epsilon dv_t \end{aligned} \tag{153}$$

where $g \in \prec [a(t), \delta a(t)]$ $h \in \prec [c(t), \delta c(t)], \underline{c}(t) \geq 0, t \geq 0$ and $\epsilon > 0$ is a small parameter (this is the case in many practical situations [9,6]).

The optimal *a priori* MS-error is bounded from above and below, perturbation methods for the bounds are used to show that the upper bound approaches the lower one as ϵ becomes smaller.

The result is quoted for h linear but holds for nonlinearities h which tend asymptotically to be linear, i.e., δc is small (see remark 2). This type of (almost linear) nonlinearities arise in practice and are usually modeled as being linear [13].

Proposition 4-1: *Assume that $\delta c = 0$ (i.e. h is linear) and $c(t) > 0$, then the optimal MS-error $p(t)$ satisfies the following*

$$p(t) = \frac{\sigma(t)}{c(t)}\epsilon + \wr(\epsilon) = E(x_t - x_t^F)^2$$

where $\lim_{\epsilon \to 0} \frac{\wr(\epsilon)}{\epsilon} = 0$ and x_t^F denotes anyone of the three asymptotically optimal filters listed below.

(F_1) *The BOF:*

$$dx_t^* = g(t, x_t^*)dt + \frac{c(t)}{\epsilon^2}u(t)[dy_t - c(t)x_t^* dt], \quad x^*(0) = 0 \tag{155}$$

$$\dot{u}(t) = \sigma^2(t) + 2\bar{a}(t)u(t) - \frac{c^2(t)}{\epsilon^2}u^2(t), \quad u(0) = \sigma_0^2 \tag{156}$$

(F_2) *The constant gain BOF (CGBOF):*

$$dx_t^c = g(t, x_t^c)dt + \frac{\sigma(t)}{\epsilon}[dy_t - cx_t^c dt], \quad x_t^c(0) = 0 \tag{157}$$

(F_3) *The linear (first approximation) BOF:*

$$dx_t^\ell = \frac{\sigma(t)}{\epsilon}[dy_t - c(t)x_t^\ell dt], \quad x^\ell(0) = 0 \tag{158}$$

Equation (154) is proven for each case separately.

Proof of (F_1): From Theorem 2-1 we get:

$$\ell(t) \leq p(t) \leq p^*(t) = E(x_t - x_t^*)^2 \leq u(t) \tag{159}$$

$$\dot{u} = \sigma^2(t) + 2\bar{a}(t)u - \frac{\underline{c}^2(t)}{\epsilon^2}u^2, \quad u(0) = \sigma_0^2 \tag{160}$$

$$\dot{\ell} = \sigma^2(t) + 2\underline{a}(t)\ell - \frac{1}{\epsilon^2}[\bar{c}^2(t) + 4\frac{\epsilon^2}{\sigma^2(t)}(\delta a)^2]\ell^2 \tag{161}$$

$$\ell(0) = \sigma_0^2$$

It can be easily seen by inspection of (160) and (161) that $u(t)$ and $\ell(t)$ are of different order in ϵ if δc is nonzero. Let's show this explicitly.

Expanding $u(t)$ as

$$u(t) \sim \sum_{n=0}^{\infty} u_n(t)\epsilon^n \tag{162}$$

yields

$$u^2(t) \sim \sum_{n=0}^{\infty} d_n\epsilon^n \tag{163}$$

$$d_n(t) = \sum_{j=0}^{n} u_j(t)u_{n-j}(t)$$

e.g.,

$$\begin{aligned} d_0(t) &= u_0^2(t) \\ d_1(t) &= 2u_0(t)u_1(t) \\ d_2(t) &= 2u_0u_2 + u_1^2 \end{aligned}$$

Plugging (162) and (163) in (160) gives:

$$\sum_{n=0}^{\infty} \dot{u}_n\epsilon^n = \sigma^2(t) + 2\bar{a}\sum_{0}^{\infty} u_n\epsilon^n - \frac{\underline{c}^2}{\epsilon^2}\sum_{0}^{\infty} d_n\epsilon^n \tag{164}$$

Equating powers of ϵ, starting with ϵ^{-2}, yields $d_0 = 0$, i.e., $u_0(t) = 0$. This in turn implies that $d_1 = 0$.

Similarly $\sigma^2 - \underline{c}^2 d_2 = 0$. But since $d_2 = u_1^2$, it follows that $u_1(t) = \frac{\sigma(t)}{\underline{c}(t)}$, i.e.,

$$u(t) = \frac{\sigma(t)}{\underline{c}(t)}\epsilon + O(\epsilon^2) forevery 0 \leq t \leq T \tag{165}$$

By a similar procedure we get $\ell_0 = 0$ and $\ell_1 = \frac{\sigma(t)}{\bar{c}(t)}$ that is

$$\ell(t) = \frac{\sigma(t)}{\bar{c}(t)}\epsilon + O(\epsilon^2) 0 \leq t \leq T \tag{166}$$

We conclude from (165) and (166) that if $\delta c = 0$, i.e., $h(t, x) = c(t)x$ then:

$$u(t) \sim \ell(t) = \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon^2) 0 \leq t \leq T \tag{167}$$

which establishes the asymptotic optimality of the BOF as $\epsilon \to 0$.

Note: These approximations are obviously not valid in the immediate vicinity of $t = 0$ where $u(0) = \ell(0) = \sigma_0^2$. This (boundary layer) problem is negligible. It can indeed be easily shown that the duration of the transient regime for this type of ode's is $O(\epsilon)$ (also see Figure 2).

This suggests the following:

(i) since $u(t) = \epsilon u_1(t) + O(\epsilon^2)$, one can replace $u(t)$ in (148) by $\epsilon u_1 = \epsilon \frac{\sigma(t)}{c(t)}$ and hope to achieve asymptotic optimality as well. The new filter clearly would have the advantage that the gain $k(t) = \frac{\sigma(t)}{\epsilon}$, thus avoiding solving a Riccati equation and therefore resulting in faster computations.

(ii) If the answer to (i) is affirmative, the next question is whether the same thing would hold for the first approximation (when expanding x_t^c) filter:

$$dx_t^\ell = \frac{\sigma(t)}{\epsilon}[dy_t - c(t)x_t^\ell dt]$$

It turns out that both filters are asymptotically optimal as is shown next.

Proof of (F_2): An upper bound on the MS-error corresponding to filters such as (F_2) can be obtained by following the first steps in the proof of Proposition 3-2 (also Section 2-2 in [21]. In this case

$$E(x_t - x_t^c)^2 \leq u^k(t)$$

where ($k(t) = \frac{\sigma(t)}{\epsilon}$):

$$\dot{u}^k = 2\sigma^2(t) + 2[\bar{a}(t) - \frac{\sigma(t)c(t)}{\epsilon}]u^k u^k(0) = \sigma_0^2 \tag{169}$$

By setting $u^k(t) \sim \sum_i u_i^k(t)\epsilon^i$ in (169), one easily obtains

$$u_0^k(t) = 0, \quad u_1^k(t) = \frac{\sigma(t)}{c(t)}$$

hence:

$$p(t) = E(x_t - x_t^c)^2 = \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon^2), \quad 0 \leq t \leq T$$

(Recall that: $p(t) \geq \ell(t) = \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon^2)$)

Proof of (F_3): Similarily, it is readily obtained that $p^\ell(t) := E[x_t - x_t^\ell]^2$ satisfies

$$\dot{p}^\ell = 2\sigma^2(t) + 2E(x_t - x_t^\ell)g(t, x_t) - 2\frac{c(t)\sigma(t)}{\epsilon}p^L$$

Using the Schwartz inequality:

$$Eab \leq E^{\frac{1}{2}}a^2.E^{\frac{1}{2}}b^2$$

and the comparison theorem (see appendix) we get $p^\ell(t) \leq u^\ell(t)$ where

$$\dot{u}^\ell = 2\sigma^2(t) + 2\theta(t)(u^\ell)^{\frac{1}{2}} - 2\frac{c(t)\sigma(t)}{\epsilon}u^L \tag{170}$$

with $\theta(t) = E^{\frac{1}{2}}g^2(t, x_t)$. Expanding $u^\ell \sim \sum_0^\infty u_i^\ell \epsilon^{\frac{i}{2}}$ in (170) and equating powers of ϵ gives $u_0^\ell = u_1^\ell = 0$ and $u_2^\ell = \frac{\sigma(t)}{c(t)}$, hence,

$$u^\ell(t) = \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon^{\frac{3}{2}}), 0 \leq t \leq T$$

Therefore,

$$p(t) = p^\ell(t) = \frac{\sigma(t)}{c(t)}\epsilon + \wr(\epsilon), \quad 0 \leq t \leq T$$

QED

Remark (1): (i) If $\sigma(t) = \sigma$ and $c(t) = c$ then $\dot{\psi}_1(t) = \dot{\ell}_1(t) = 0$ and the next terms in the expansion of $u(t)$ and $\ell(t)$ are:

$$u_2(t) = \frac{1}{c^2}\bar{a}(t)$$
$$\ell_2(t) = \frac{1}{c^2}\underline{a}(t)$$

so that $u(t) = \ell(t) + O(\epsilon^3)$ if and only if $\delta a = 0$, i.e., both g and h are linear.

(ii) In [14], it was shown that for incrementally conic nonlinearities we have the following lower bound $\ell(t)$:

$$p(t) \geq \ell(t) = (1 - s(t))r(t) \tag{172}$$

where $s(t)$ is the unique nonnegative root of

$$(1 - s(t))e^{s}(t) = e^{-d(t)} \tag{173}$$

$$d(t) = \int_0^t [\frac{\delta a(s)}{\sigma^2(s)} + \frac{\delta c^2(s)}{\epsilon^2}]q(s)ds \tag{174}$$

$$\dot{q} = \sigma^2(t) + \frac{c^2(t)}{\epsilon^2}r^2(t) + 2[\bar{a}(t) - \frac{c^2(t)}{\epsilon^2}r]q \tag{175}$$

$$q(0) = \sigma_0^2$$

$$\dot{r} = \sigma^2(t) + 2a(t)r - \frac{c^2(t)}{\epsilon^2}r^2, \quad r(0) = \sigma_0^2 \tag{176}$$

From (160) and (165) we readily get that $r(t) = \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon^2)$. It is therefore clear from (172) that if $s(t) = O(\epsilon)$, then $\ell(t) = \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon^2)$ the same as the one we have used here.

This is indeed the case: (175) implies $q(t) = O(\epsilon)$ and (174) that $d(t) = O(\epsilon)(\delta c = 0)$. Assuming $s(t) \sim \sum_0^\infty s_n \epsilon^n$ and letting ϵ go to zero in (47) gives that $1 - s_0 = e^{-s_0}$ necessarily. This has the unique solution $s_0 = 0$, hence $s(t) = O(\epsilon)$.

Remark (2): Almost linear observations.

The same results in previous proposition can be extended to the particular class of nonlinearities $h \in\prec [c, \delta c]$ where δc is also a small parameter. Indeed, the upper and lower bounds u and ℓ on $p(t)$ and $p^*(t) := E(x_t - x_t^*)^2$ where x_t^* is the BOF in (F_1) (with cx_t^* and c replaced by $h(x_t^*)$ and $\bar{c}$) are given by (165) and (166):

$$u(t) = \frac{\sigma(t)}{\underline{c}(t)}\epsilon + O(\epsilon^2)$$

$$= \frac{\sigma(t)}{c(t)}\epsilon(1 + \frac{\delta c}{c} + O((\delta c)^2)) + O(\epsilon^2)$$

$$= \frac{\sigma(t)}{c(t)}\epsilon + \frac{1}{c}\epsilon\delta c + O(\epsilon^2) + \epsilon O((\delta c)^2)$$

Thus, for small δc

$$u(t) = \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon, \delta c)$$

Similarly

$$\ell(t) = \frac{\sigma(t)}{\bar{c}(t)}\epsilon + O(\epsilon^2)$$

$$= \frac{\sigma(t)}{c(t)}\epsilon(1 - \frac{\delta c}{c} + O((\delta c)^2)) + O(\epsilon^2)$$

$$= \frac{\sigma(t)}{c(t)}\epsilon + O(\epsilon, \delta c))$$

It is not hard either to establish that for the analogs of the filters (F_2) and (F_3) (as in (157) and (158), but with cx replaced by $h(x)$) the upper bounds are

$$u^k(t) = \frac{\sigma(t)}{\underline{c}(t)}\epsilon + O(\epsilon^2)$$

and

$$u^\ell(t) = \frac{r(t)}{\underline{c}(t)}\epsilon + O(\epsilon^{3over2})$$

which makes these filters asymptotically optimal as δc and ϵ become smaller with

$$p(t) = \frac{\sigma(t)}{c(t)}\epsilon + \wr(\epsilon).$$

Application to the Benes filter Let

$$dx_t = f(x_t)dt + dw_t \tag{177}$$

$$dy_t = x_t dt + dv_t \tag{178}$$

where the drift f satisfies

$$f_x(x) + f^2(x) = ax^2 + bx + c \tag{179}$$

with $a \geq 0$ to prevent finite time escape situations.

As mentioned earlier, this is one of the few nonlinear filtering problems which was shown to admit a finite number of sufficient statistics [3]. We are interested here in investigating this class of filtering problems when the diffusion process $\{x_t\}$ is measured in a low noise channel. In particular, we would like to know what type of implementation simplifications will result from this additional assumption. Accordingly, let $\{x_t\}$ be as in (177) and:

$$dy_t = x_t dt + \epsilon dv_t \tag{180}$$

In order to know how ϵ enters Benes' original formulas, we shall transform the DMZ equation in Fisk-Stratonovich form by following the steps outlined below. The unnormalized pdf $u(t,x)$ satisfies the following stochastic PDE:

$$du = (\mathcal{L}^*(u) - \frac{1}{2}\frac{x^2}{\epsilon^2}u)dt + \frac{x}{\epsilon^2}udy \tag{181}$$

$$\mathcal{L}^*(u) = \frac{1}{2}u_{xx} - (fu)_x$$

which in our case is

$$du = [\frac{1}{2}u_{xx} - fu_x - (f_x + \frac{1}{2}\frac{x^2}{\epsilon^2})u]dt + \frac{x^2}{\epsilon^2}udy \tag{182}$$

By letting $V(t,x) = e^{-\frac{1}{\epsilon^2}xy_t}u(t,x)$, the stochastic differentials in (182) are eliminated. We obtain the following classical PDE (robust DMZ):

$$V_t = \frac{1}{2}V_{xx} + (\frac{y_t}{\epsilon^2} - f)V_x - (\frac{y_t}{\epsilon^2}f + f_x + \frac{1}{2}\frac{x^2}{\epsilon^2} - \frac{1}{2}\frac{y_t^2}{\epsilon^4})V \tag{183}$$

Using $V(t,x) = e^{\int_0^x f(\sigma)d\sigma}\rho(t,x)$ and (179) in (183), we get after some computations that:

$$\rho_t = \frac{1}{2}\rho_{xx} + \frac{1}{2}y_t\rho_x + [\frac{1}{2}\frac{y_t^2}{\epsilon^4} - \frac{1}{2}\frac{1}{\epsilon^2}(1+\epsilon^2 a)x^2 - \frac{1}{2}bx - \frac{1}{2}c]\rho$$

It can be easily verified that ρ is given by

$$\rho(t,x) = \exp\left\{-\frac{(x-mu_t)^2}{2\theta(t)}\right\}$$

where

$$\begin{aligned} \dot\theta(t) &= 1 - \frac{1}{\epsilon^2}(1+\epsilon^2 a)\theta^2(t), \quad \theta(0)=0 \\ d\mu_t &= -\frac{1}{\epsilon^2}(1+\epsilon^2 a)\theta(t)\mu_t dt - \frac{1}{2}\theta(t)bdt + \frac{1}{\epsilon^2}\theta(t)dy_t \end{aligned} \tag{184}$$

$$u(t,x) = e^{\frac{1}{\epsilon^2}xy_t}\exp\left\{\int_0^x f(\sigma)d\sigma - \frac{1}{2}\frac{(x-\mu_t)^2}{\sigma(t)}\right\} \tag{185}$$

Our goal is to see under what circumstances μ_t can be a good approximation for the conditional mean $E(x_t|\mathcal{Y}_0^t)$ given by:

$$E[x_t|\mathcal{Y}_0^t] = \int x\frac{u(t,x)}{\int u(t,x)}dx$$

It turns out that for cone bounded drifts in (179) (e.g., $f(x) = \tanh(x)$ or linear), the following holds.

Claim: $\{\mu_t\}$ is asymptotically optimal as $\epsilon \to 0$

To see this, rewrite (184) in the more suggestive form:

$$d\mu_t = \frac{\theta(t)}{\epsilon^2}[dy_t - (1+\epsilon^2 a)\mu_t dt] - \frac{1}{2}\theta(t)bdt$$

and notice that $\theta(t) = \frac{\epsilon}{(1+\epsilon^2 a)^{\frac{1}{2}}}\tanh((1+\epsilon^2 a)\frac{t}{\epsilon}) \sim \epsilon + O(\epsilon^3)$. It is not hard then to show that $\mu_t = \mu_t^\ell + O(\epsilon)$ where

$$d\mu_t^\ell := \frac{1}{\epsilon}[dy_t - \mu_t^\ell dt]$$

is precisely the linear BOF obtained in last proposition which was shown to be asymptotically optimal as ϵ becomes smaller.

Notice that for the particular case $f(x) = \tanh(x)$, $a = b = 0$ and $c = 1$ and hence

$$d\mu_t = \frac{\tanh(t/\epsilon)}{\epsilon}[dy_t - \mu_t dt]$$

2.5 Examples and Simulation Results

Example 1: In this example, the asymptotic optimality of "KF" for WNL systems is illustrated. We consider:

$$\begin{aligned} dx_t &= ax_t dt + \epsilon \tanh(x_t)dt + \sigma dw_t \\ dy_t &= cx_t dt + \rho dv_t \end{aligned}$$

$$x_0 \sim \mathcal{N}(m_0, \sigma_0^2)$$

where $f(\cdot) = \tanh(\cdot) \in \prec [\frac{1}{2}, \frac{1}{2}]$, i.e., $\underline{\mu} = 0, \bar{\mu} = 1, \mu = \Delta\mu = \frac{1}{2}$.

Simulations were done using a Monte Carlo technique and the following numerical data:

$$a = -1, \sigma = \rho = 0.3, c = 1, m_0 = 0, \sigma_0 = 0.1$$

The results are summarized in the plots of Figures 3,4, and 5. which correspond to different values of ϵ ($\epsilon = 0.2, 0.1$ and 0.05 respectively). In each figure, we have plotted $p^k(t) := E(x_t - x_t^k)^2$, $r(t)$ and $\ell(t)$; the latter being the lower bound on the optimal MS-error $p(t)$ which therefore lies between $\ell(t)$ and $p^k(t)$.

The plots appear to corroborate the results of Proposition 3-2 in which it is stated that the "KF" is asymptotically optimal as ϵ becomes smaller and that $r(t)$ is a good approximation for the (unknown) optimal MS-error $p(t)$ in the sense that $p^k(t) \sim p(t) = r(t) + O(\epsilon)$.

Example 2: This second example deals with the asymptotic optimality of the BOF and CGBOF in the case of low measurement noise level filtering

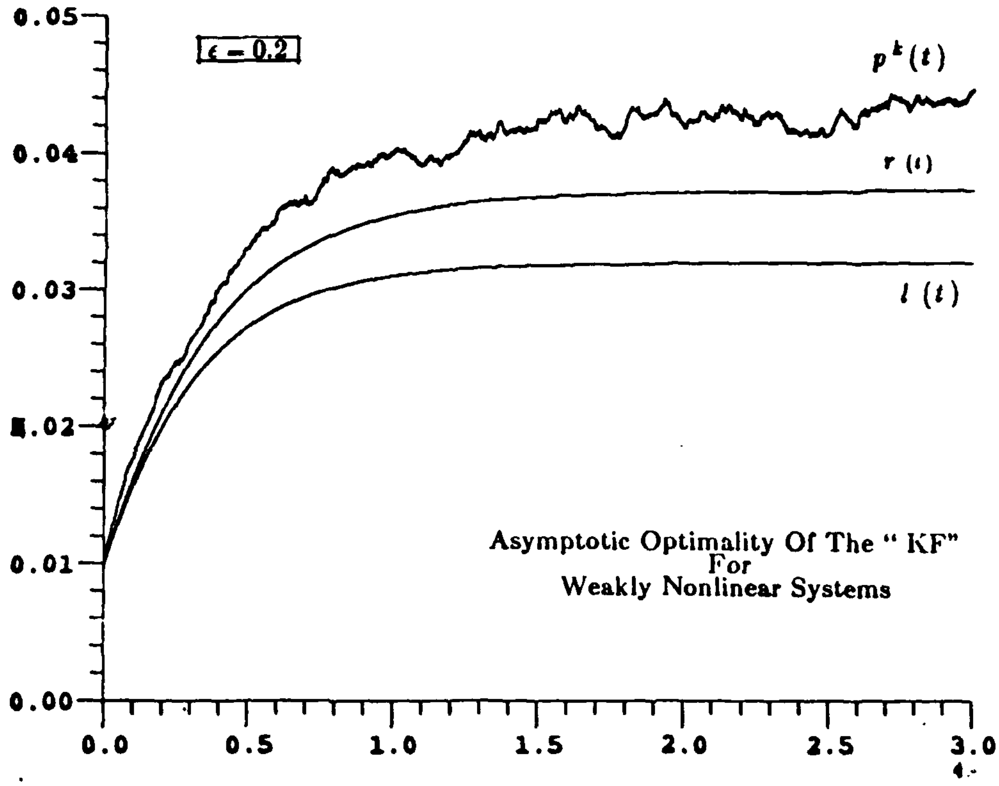

Figure 8: KF performance $\epsilon = 0.2$.

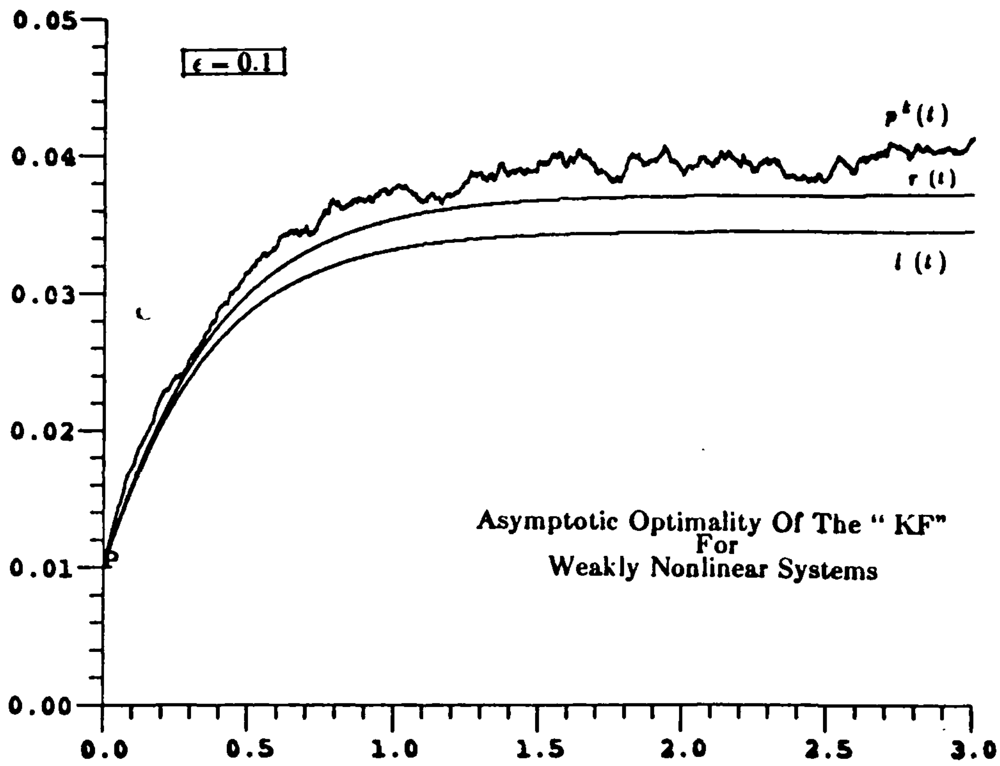

Figure 4: KF performance $\epsilon = 0.1$.

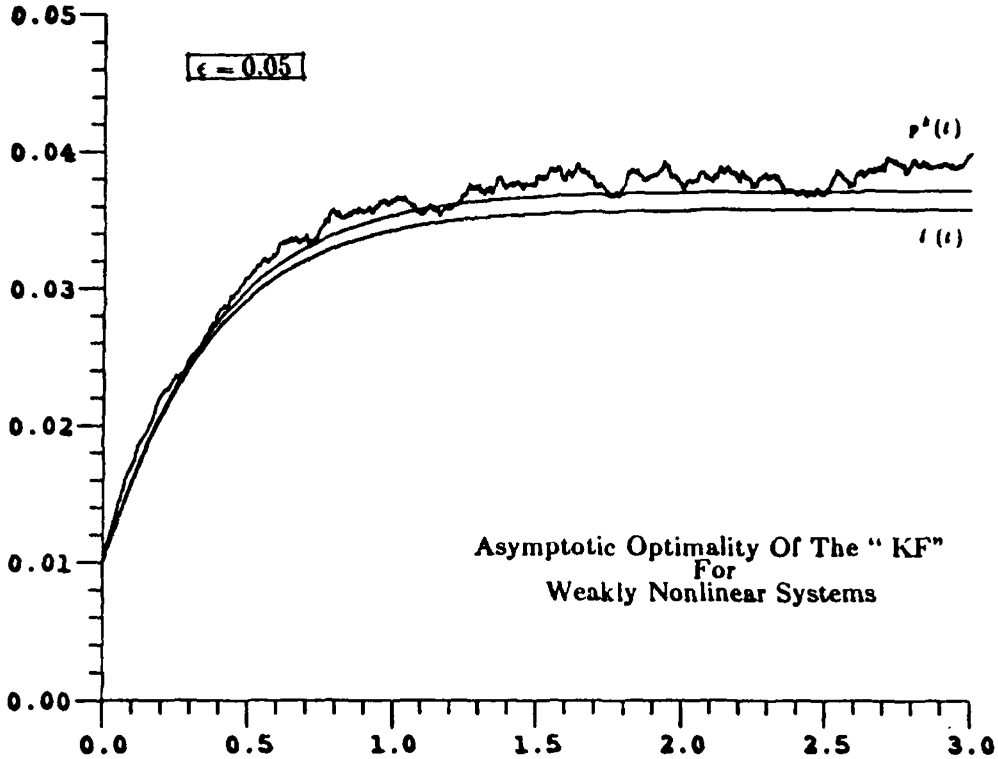

Figure 5: KF performance $\epsilon = 0.2$.

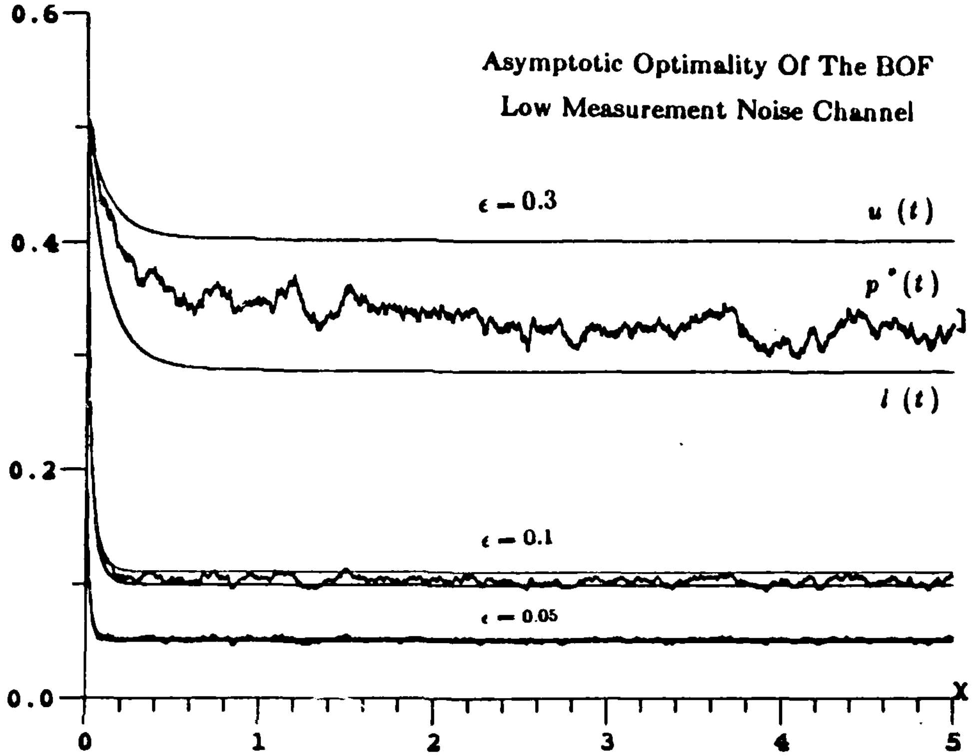

Figure 6: BOF performance

problems. The following model is considered:

$$\begin{aligned} dx_t &= \arctan(x_t)dt + \sigma dw_t \\ dy_t &= cx_t dt + \epsilon dv_t \\ x_0 &\sim \mathcal{N}(m_0, \sigma_0^2) \end{aligned} \tag{186}$$

where $g(\cdot) = \arctan(\cdot) \in\prec [\frac{1}{2}, \frac{1}{2}]$, i.e., $a = \delta a = \frac{1}{2}$ and

$$a = -1, \sigma = c = 1, m_0 = 0, \sigma_0^2 = 0.5$$

The simulations are summarized in Figures 6 and 7 which correspond to the performance of the BOF and CGBOF respectively. Each figure contains 3 sets of plots corresponding to $\epsilon = 0.3, 0.1$ and 0.05 from top to bottom. Each set of 3 curves consist of the upper bound $u(t)$ on the BOF, the MS-error $p^F(t) = E(x_t - x_t^F)^2$ and the lower bound $\ell(t)$ on the optimal MS-error $p(t)$.

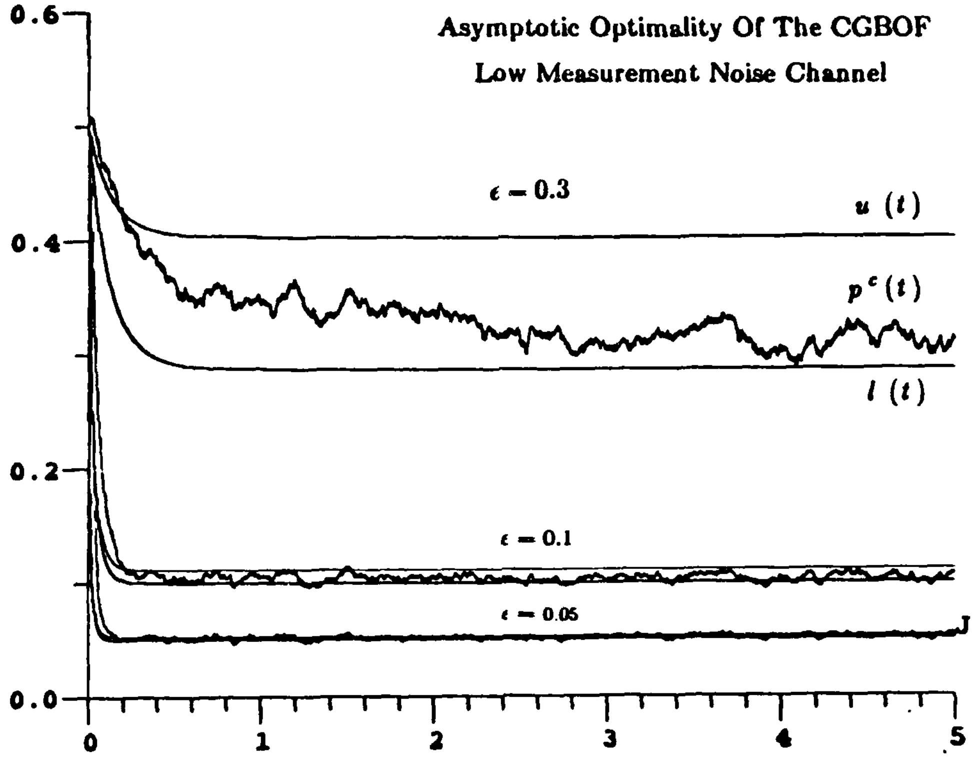

Figure 7: CGBOF performance

Again, these plots seem to agree with the results of Proposition 4-1 in which it is stated that the BOF and CGBOF are both asymptotically optimal as ϵ becomes smaller and that $\frac{\sigma(t)}{c(t)}\epsilon$ (equal to ϵ here) is a good approximation for the (unknown) optimal MS-error $p(t)$.

Remark: It can be seen in Figure 2(b) that the MS-error $p^c(t)$ exceeds the (BOF) upper bound $u(t)$ in all three cases as might be expected. To see why this is so, it suffices to recall that the CGBOF was obtained by approximating the BOF gain $k^*(t) := \frac{c(t)}{\epsilon^2}u(t)$ by $\frac{\sigma(t)}{\epsilon}$ since $u(t) \sim \frac{\sigma(t)}{c(t)}\epsilon$. However, it was remarked earlier that this last approximation does not hold in the immediate vicinity of $t = 0$ (boundary layer problem). Outside this region (which shrinks to zero as $\epsilon \to 0$), the CGBOF performs in a comparable fashion than the BOF with the speed advantage.

2.6 Conclusions

We investigated the asymptotic behavior question of one dimensional nonlinear filtering problems involving drifts with bounded derivatives using an upper and lower bound approach to show that the *a priori* mean square error associated with some suboptimal filters approaches the optimal one asymptotically. This approach demonstrates that significant information can be infered from the derivative bounds (i.e., of the cone in which the nonlinearities reside). In particular, it is shown that in the case of weakly nonlinear systems, that the "KF" (designed for the underlying linear system) is asymptotically optimal as $\epsilon \to 0$. In other words the nonlinearity can be ignored as long as the asymptotic behavior is concerned.

In the case of diffusions measured in a low noise channel, three asymptotically optimal filters were obtained, one of which is linear. Furthermore, asymptotic values for the unknown optimal MS-error were obtained in both cases.

The main point is that upper and lower bounds on the optimal MS-error, when available, may be used (in addition to permance testing of suboptimal designs) as a relatively simple tool to study certain nonlinear filtering problems.

Appendix

Theorem (1): (Comparison Theorem [15]) *Let $F(x,y)$ and $G(x,y)$ be continuous in the rectangle*

$$D : |x - x_0| < a, \quad |y - y_0| < b$$

and suppose that $F(x,y) < G(x,y)$ everywhere in D. Let $y(x)$ and $z(x)$ be the solutions of

$$\begin{aligned} \dot{y} &= F(x,y), \quad y(x_0) = \alpha \\ \dot{z} &= G(x,y), \quad z(x_0) = \alpha \end{aligned} \tag{187}$$

Let I be the largest subinterval of $(x_0 - a, x_0 + a)$ where both $y(x)$ and $z(x)$ are defined and continuous; then for $x \in I$

$$z(x) < y(x), \quad x < x_0$$

$$z(x) > y(x), \quad x > x_0$$

Theorem (2): (Perron [12]) *If $F(t), f_i(t)$, $\quad t_0 \in [0, \infty[, i = 1, \ldots, n$, are real continuous functions of t having finite limits*

$$\lim_{t \to \infty} F(t) = b, \lim_{t \to \infty} f_i = a_i,$$

if the roots $\lambda_i, i = 1, \ldots, n$ of the equation

$$\rho^n + a_1 \rho^{n-1} + \ldots + a_n = 0$$

are real, distinct, and different from 0, then the equation

$$\frac{d^n}{dt^n} y(t) + f_1(t) \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + f_n(t) y(t) = F(t) \tag{188}$$

has at least one solution $y(t)$ with $\lim_{t \to \infty} y(t) = \frac{b}{a_n}$, $\lim_{t \to \infty} d^m/dt^m y(t) = 0$. If $\lambda_i < 0, i = 1, \ldots, n$, then all solutions of (188) have these properties.

References

[1] Arnold, L., "Stochastic Differential Equations," Wiley, New York, 1974.

[2] Baras, J., Blankenship, G.L., Mitter, S.K, "Nonlinear Filtering of Diffusion Processes," IFAC, Triennial World Congress, Kyoto, Japan, 1981.

[3] Benes, V.E., "Exact Finite Dimensional Filters for Certain Diffusions with Non-Linear Drifts," Stochastics, 1981.

[4] Blankenship, G.L., "Asymptotic Analysis in Mathematical Physics and Control Theory, Some Problems with Common Features," Ricerche di Automatica, Vol. 10, 1979.

[5] Blankenship, G.L., "Some Approximation Methods in Nonlinear Filtering," IEEE Control And Decision Proc., 1980, pp. 51-56.

[6] Bobrovsky, B.Z., Geering, H.P., "Simple and Efficient Linear and Nonlinear Filters by Regular Perturbation Methods," Balakrishnan, A.V. and Thoma, M. (eds), Stochastic Differential Systems, Springer Verlag, 1982.

[7] Bobrovsky, B.Z., Zakai, M., "A Lower Bound on the Estimation Error for Certain Diffusion Processes," IEEE Trans. on Inf. Theory, Vol. 92, 1976.

[8] Bobrovsky, B.Z., Zakai, M., "Asymptotic *A Priori* Estimates for the Error in the Nonlinear Filtering Problem," IEEE, Trans. on Inf. Theory, Vol 28, March 1982.

[9] Bobrovsky, B.Z., Zakai, M., "Asymptotic Bounds on the Minimal Error of Non Linear Filtering," Hazewinkel, M. and Willems, J.C. (eds), Stochastic Systems: the Mathematics of Filtering and Identification and Applications, D. Reidel, Amsterdam, 1983, pp. 573-581.

[10] Brockett, R.W., "Asymptotically Optimal Estimation," IEEE Decision and Control Conference, 1981.

[11] Bucy, R.S, Joseph, P.D, "Filtering for Stochastic Processes with Application to Guidance," New York, Interscience Publishers, 1968.

[12] Cesari, L., "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations," Springer Verlag, 1971.

[13] Gilman, A.S., Rhodes, I.B., "Cone Bounded Nonlinearities and Mean Square Bounds - Estimation Upper Bounds," IEEE A.C., Vol 18, 1973, pp. 260-265.

[14] Gilman, A.S., Rhodes, I.B., "Cone Bounded Nonlinearities and Mean Square Bounds - Estimation Lower Bounds," IEEE A.C., 1976.

[15] Hille, E., "Lecture notes on ODE's," Addison Wesley, 1965.

[16] Jazwinski, A.H., "Stochastic Processes and Filtering Theory," Academic Press, New York, 1970.

[17] Kalman, R.E., "A New Approach to Linear Filtering and Prediction Problem " J. Basic Eng., Trans. ASME, Series D, Vol. 82, 1960.

[18] Kalman, R.E., Bucy, R.S., "New Results in Filtering and Prediction theory," 1961, J. Basic Eng, Trans AMSE, Series D, Vol 83, pp. 95-107.

[19] Katzur, R., Bobrovsky, B.Z, Schuss, Z., "Asymptotic Analysis of the Optimal Filtering Problem for One Dimensional Diffusions Measured in a Low Noise Channel, Part 1," SIAM J of Appl Math., 1986.

[20] Saydy, L., "Asymptotic Behavior Of Nonlinear Filters" M.S. Thesis, University of Maryland, August 1985.

[21] Saydy, L., Blankenship, G.L., "A Bound Approach To Asymptotic Analysis In Some Nonlinear Filtering Problems," Submitted.

[22] Saydy, L., Blankenship, G.L., "Optimal Stationary Behavior In Some Nonlinear Stochastic Filtering Problems - A Bound Approach," submitted for publication.

[23] Wong, W.S., "New Classes of Finite Dimensional Nonlinear Filters," Systems and Control Letters, 3, 1983, 155-164.

3 Optimal Sensor Scheduling in Nonlinear Filtering

3.1 Introduction

3.1.1 Motivation and preliminaries

The problem of nonlinear filtering of diffusion processes has received considerable attention in recent years; see the anthologies [1], [2], [3] for a review of important developments. In current studies as well as in related analyses of the partially observed stochastic control problem with such models [4], [5], a key role is played by the linear stochastic partial differential equation describing the evolution of the unnormalized conditional probability measure of the state process given the past of the observations, the so called Zakai equation.

A significant byproduct of these advances is the feasibility of analyzing complex signal processing problems, including adaptive and sensitivity studies, in an integrated, systematic manner, without heuristic or *adhoc* assumptions. A problem of interest in this area is the so called *sensor scheduling problem*. Roughly speaking this problem is concerned with the simultaneous selection (according to some performance measure) of a signal processing scheme *together* with the sensors that collect the data to be processed. Particular applications include multiple sensor platforms, distributed sensor networks, large scale systems. For example, in a multiple sensor platform, there is definite need for coordinating the data obtained from the various sensors which may include radar, infrared, sonar, etc. The data obtained from different sensors are of varying quality and a systematic way is needed for allocating confidence or basing decisions on data collected from different types of sensors. For example radar sensors are more accurate than infrared sensors for long range tracking while the opposite is true for short range tracking. In sensor networks one needs to coordinate data collected from a large number of sensors distributed over a large geographical area. Conflicts should be resolved and a preferred set of sensors must be selected, over finite (short) time intervals, and utilized in detection, estimation or control decisions. Similarly in large scale systems there is typically an attached information network with the objective of collecting data, processing them and making the results available to the many control agents for their decisions (actions). Again the need for coordinating this information in a systematic way is critical.

In such sensor scheduling problems the systematic utilization of sensors should be the result of optimizing reasonably defined performance measures. Clearly these performance measures shall include terms allocating penalties for errors in detection and/or estimation. But more importantly, they must include terms for costs associated with turning sensors on or off, and for switching from one sensor to another. Examples of such costs arising in practice abound. Turning on a radar sensor increases the detectability of the platform (since radars are active sensors) and this should be reflected as a switching cost. Deciding to use a more accurate, albeit more complex sensor, will require higher bandwidth communications and often more computational power allocated to that sensor. In distributed sensor networks it may mean the physical movement of a sensor carrying platform (such as a helicopter or airplane) to a particular geographical location. In large scale systems the utilization of several (often hundreds) sensors for decision making may provide better average performance but it certainly reduces the response speed of the system to changing conditions, and it increases computational and communication costs both in terms of hardware and software. The latter are obviously evident in large computer/communication networks. These running

and swithching costs will depend often on the part of the state space occupied by the state vector, i. e. they will be functions of the state as well. For example sensors have different accuracy or noise characteristics when the state process takes values in different areas of the state space. Also there is cost associated with handling the transfer of information, or tracking record, when there are changes in the set of sensors used; and these costs often depend on the state process.

It is not our intent to provide an extensive description of applications here. Detailed descriptions of some of these problems can be found elsewhere; see for example [6], [7]. The underlying thread in all these problem areas is the existence of a variety of sensors, which provide data (for processing) including information of widely varying quality about parameters or variables of interest, for control, detection, estimation etc. Due to the complexity of these problems it is important to develop systematic conceptual, analytical and numerical methods for their study and to reduce reliance on ad hoc, heuristic methods as much as possible. The present paper is offered as a contribution in this direction. It provides a general methodology to this problem by reducing it to the analysis of a system of quasi-variational inequalities (see section 3 for details). Numerical methods will be described elsewhere [13].

The sensor scheduling problem is considered here in the context of non-linear filtering of diffusion processes, and is therefore applicable to detection problems with the same signal models. Modifications of the results apply to other situations including control. In the next section we present a somewhat heuristic definition of the problem, intended to describe the problem clearly, at an intuitive level. The intricasies of establishing this model in a rigorous mathematical fashion are given in section 2, and constitute one of the main contributions of the paper.

3.1.2 Preliminary description of the problem

The problem considered is as follows. A signal (or state) process $x(\cdot)$ is given, modelled by the diffusion

$$\begin{aligned} dx(t) &= f(x(t))dt + g(x(t))dw(t) \\ x(0) &= \xi \end{aligned} \tag{1.1}$$

in $\mathbb{R}^n$. We further consider M noisy observations of $x(\cdot)$, described by

$$\begin{aligned} dy^i(t) &= h^i(x(t))dt + R_i^{1/2}dv^i(t), \\ y^i(0) &= 0 \end{aligned} \tag{1.2}$$

with values in $\mathbb{R}^{d_i}$. Here $w(\cdot)$, $v^i(\cdot)$ are independent, standard, Wiener processes in $\mathbb{R}^n$, $\mathbb{R}^{d_i}$ respectively, and $R_i = R_i^T > 0$ are $d_i \times d_i$ matrices. Further mathematical details on the system (1.1), (1.2) will be given in section 2. Let us consider a finite time horizon $[O,T]$. To formulate the problem of determining an *optimal utilization schedule* for the available sensors, so as to *simultaneously minimize* the cost of errors in estimating a function of $x(\cdot)$ and the costs of using as well as of switching between various sensors, we need to specify these costs. To this end, let $c_i(x)$ denote the cost per unit time when using sensor i, and the state of the system is x; $k_{io}(x)$, $k_{oi}(x)$ denote the cost for turning off, respectively on, the ith sensor when

the state of the system is x. The objective of the performed signal processing is to compute, at time T, an estimate $\hat{\phi}(T)$ of a given function $\phi(x(T))$ of the state. Penalties for errors in estimation are assesed according to the cost function

$$E\{c_e(\phi(x(T)) - \hat{\phi}(T))\} := E\{|\phi(x(T)) - \hat{\phi}(T)|^2\} \tag{1.3}$$

We shall comment briefly on more general estimation problems in section 4 of this paper. In particular the consideration of a quadratric $c_e(\cdot)$ is not a serious restriction.

We consider next, the set of all possible *sensor activation configurations*, denoted here by $\mathcal{N}$. An element $\nu \in \mathcal{N}$ is a *word* of length M from the alphabet $\{0,1\}$. If the ℓ^{th} position is occupied by an 1, the ℓth sensor is activated (used), if by a 0 the ℓ^{th} sensor is off. There are $N = 2^M$ elements in $\mathcal{N}$. *A schedule of sensors is then a piecewise constant function* $u(\cdot) : [O,T] \to \mathcal{N}$. We let $\tau_j \in [O,T]$ denote the instants of changing schedule; i. e. , the moments when at least one sensor is turned on or off. At such a switching moment, suppose the schedule before is characterized by $\nu \in \mathcal{N}$, and after by $\nu' \in \mathcal{N}$. Then the *switching cost associated with such a scheduling change* will be

$$\mathbf{k}_{\nu\nu'}(x) := \sum_{\{i\in\nu\}\{i\notin\nu'\}} k_{io}(x) + \sum_{\{j\notin\nu\}\{j\in\nu'\}} k_{oj}(x). \tag{1.4}$$

The *total running cost, associated* with schedule $\nu \in \mathcal{N}$ will be

$$\mathbf{c}_\nu(x) := \sum_{\{j\in\nu\}} c_j(x) \tag{1.5}$$

In (1.4), (1.5), *the symbol* $\{i \in \nu\}$ *denotes the set of all indices (from the set* $\{1, 2, \ldots, M\}$ *which are occupied by an 1 in* ν (i. e. the indices corresponding to the sensors which are on); similarly the symbol $\{i \notin \nu\}$ *denotes the set of indices corresponding* to sensors that are off.

Using the above notation the available observations, *under sensor schedule* $u(\cdot)$ are described by

$$dy(t, u(t)) := h(x(t), u(t))dt + r(u(t))dv(t), \tag{1.6}$$

where it is apparent that the available observations depend explicitly on the sensor schedule $u(\cdot)$. In (1.6), for $x \in \mathbb{R}^n$, $\nu \in \mathcal{N}$,

$$h(x,\nu) := \begin{bmatrix} h^1(x)\chi_{\{\nu\}}(1) \\ \vdots \\ h^i(x)\chi_{\{\nu\}}(i) \\ \vdots \\ h^M(x)\chi_{\{\nu\}}(M) \end{bmatrix}, \tag{1.7}$$

a block column vector, where in standard notation

$$\chi_{\{\nu\}}(i) := \left\{ \begin{array}{ll} 1, & \text{if the } i^{th} \text{ position in the word } \nu \text{ is occupied by an 1} \\ 0, & \text{otherwise} \end{array} \right\} \tag{1.8}$$

Similarly for $\nu \in \mathcal{N}$

$$r(\nu) := \text{Block diagonal}\{R_i^{1/2}\chi_{\{\nu\}}(i)\}, \tag{1.9}$$

where R_i are the symmetric, positive matrices defined above. Finally

$$v(t) := \begin{bmatrix} v^1(t) \\ \vdots \\ v^M(t) \end{bmatrix} \tag{1.10}$$

is a higher dimensional standard Wiener process. In view of (1.7), for all $\nu \in \mathcal{N}$

$$h(\cdot,\nu) : \mathbb{R}^n \to \mathbb{R}^D, \tag{1.11}$$

while

$$r(\nu) : \mathbb{R}^D \to \mathbb{R}^D, \tag{1.12}$$

where

$$D = d_1 + d_2 + \cdots + d_M. \tag{1.13}$$

To make the notation clearer, consider the case M = 2, N = 4. Then $\mathcal{N} = \{00, 01, 10, 11\}$ and

$$\begin{aligned} h(x,00) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ h(x,01) &= \begin{bmatrix} 0 \\ h^2(x) \end{bmatrix} \\ h(x,10) &= \begin{bmatrix} h^1(x) \\ 0 \end{bmatrix} \\ h(x,11) &= \begin{bmatrix} h^1(x) \\ h^2(x) \end{bmatrix}, \end{aligned} \tag{1.14}$$

while

$$\begin{aligned} r(00) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ r(10) &= \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \\ r(01) &= \begin{bmatrix} 0 & 0 \\ 0 & R_2^{1/2} \end{bmatrix} \\ r(11) &= \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & R_2^{1/2} \end{bmatrix} \end{aligned} \tag{1.15}$$

Clearly the dimension of the range space of $y(\cdot,\nu)$ is

$$D_\nu := \sum_{i=1}^{M} d_i\, \chi_{\{\nu\}}(i). \tag{1.16}$$

Of course for all ν, $y(t,\nu) \in \mathbb{R}^D$.

Following established terminology (c.f. [9]) we see that a sensor scheduling strategy is defined by an increasing sequence of switching times $\tau_j \in [O,T]$ and the corresponding sequence $\nu_j \in \mathcal{N}$ of sensor activation configurations. We shall denote such a strategy by $u(\cdot)$, where

$$u(t) = \nu_j, \quad t \in [\tau_j, \tau_{j+1}); \quad j = 1,2,\ldots \tag{1.17}$$

As stated earlier we are interested in the *simultaneous* minimization of costs due to estimation errors as well as sensor scheduling. We shall therefore consider *joint estimation and sensor scheduling strategies.* Such a strategy consists of two parts: the sensor scheduling strategy u (see (1.17)) and the estimator $\hat{\phi}$. The set of admissible strategies U_{ad} is the customary set of strategies adapted to the sequence of σ-algebras

$$\mathcal{F}_t^{y(\cdot,u(\cdot))} := \sigma\{y(s,u(\cdot)),\ s \leq t\}. \tag{1.18}$$

That is, we consider *strict sense* admissible controls in the sense of [4]. For the problem under investigation this last statement must be interpreted very carefully. First, we have indicated in (1.18), that the available past observation data information σ-algebra depends (as is evident from (1.6) - (1.9)) very strongly on the sensor schedule $u(\cdot)$. This dependence is non-standard, as here the dimension of the observation vector and the noise covariance change drastically at each switching time τ_i. In standard stochastic control formulations [4], [5], the dependence of y on $u(\cdot)$ is much more implicit. This is a difficult part of the formulation here, since it prevents us from using Girsanov transformations in a straightforward manner. Secondly (1.18) means that the switching times τ_i and the variables ν_i, which define $u(\cdot)$, must be adapted to the filtration $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, *which depends essentially* on the values of τ_i and ν_i! Finally (1.18) also means that $\hat{\phi}(T)$ must be measurable with respect to $\mathcal{F}_t^{y(\cdot,u(\cdot))}$. We shall describe a rigorous mathematical construction of such a model in section 2.

Given such a strategy the corresponding cost is

$$J(u(\cdot),\hat{\phi}) := E\{|\phi(x(T)) - \hat{\phi}(T)|^2 \tag{1.19}$$

$$+ \int_0^T c(x(t),u(t))dt \tag{1.20}$$

$$+ \sum_j k(x(t),u(\tau_{j-1}),u(\tau_j))\}. \tag{1.21}$$

Here for $x \in \mathbb{R}^n$, $\nu,\nu' \in \mathcal{N}$

$$c(x,\nu) := \mathbf{c}_\nu(x), \tag{1.22}$$

(c.f. Eq. (1.5)), and

$$k(x,\nu,\nu') = \mathbf{k}_{\nu,\nu'}(x), \tag{1.23}$$

(c.f. Eq. (1.4)).

The optimal sensor scheduling in nonlinear filtering is thus formulated as the determination of a strategy achieving

$$\inf_{u(\cdot),\hat{\phi}} J(u(\cdot),\hat{\phi}) \tag{1.24}$$

among all admissible strategies.

To simplify the notation a little, let us order the elements of $\mathcal{N}$ according to the numbers they represent in binary form. For example in the case M = 2, N = 4 we replace $\mathcal{N} = \{00, 01, 10, 11\}$ by the set of integers $\{1, 2, 3, 4\}$. That is the one-one correspondence between $\mathcal{N}$ and $\{1, 2, \ldots, N\}$ is described by

$$\begin{aligned} \nu &\longmapsto \text{(integer represented by } \nu) + 1 \\ k &\longmapsto \text{ binary representation of } (k-1). \end{aligned} \tag{1.25}$$

So in the sequel of the paper we replace all the ν, ν' in equations (1.4) - (1.23) by the corresponding integers from {1, 2, ..., N }.

The structure of the paper is as follows. In section 2 a precise mathematical formulation is given and the corresponding stochastic control problem is precisely defined. In section 3 the set of quasi-variational inequalities solving the problem is derived. In section 4 we offer some comments and discussion for extensions, further developments and computational methods.

3.2 The Stochastic Control Formulation

3.2.1 Setting of the model

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, on which a filtration $\mathcal{F}_t$ is given, $\mathcal{A} = \mathcal{F}_\infty$. Let $w(\cdot)$ and $z(\cdot)$ be two independent, standard $\mathcal{F}_t$-Wiener processes with values in $\mathbb{R}^n$ and $\mathbb{R}^D$ respectively, carried by this probability space. On the same space we consider also an $\mathbb{R}^n$-valued random variable ξ, independent of $w(\cdot), z(\cdot)$, and with probability distribution function π_0.

We consider the Itô equation (1.1), where $f(\cdot)$ is $\mathbb{R}^n$-valued, bounded and Lipschitz, while $g(\cdot)$ is $\mathbb{R}^{n\times n}$-valued, bounded and Lipschitz. Letting $a = \frac{1}{2}gg^T$, we assume $a > \alpha I_n$, where $\alpha > 0$ and I_n is the $n \times n$ identity matrix. The Lipschitz property is unnecessary and can be easily removed using Girsanov's transformation (i.e. consider weak solutions of (1.1)) [8]. It is assumed here to simplify the technicalities not related with the main issues of the paper. Under these assumptions (1.1) has a strong solution with well known properties [8]. Note that *under* P, $z(\cdot)$ *is independent of* $x(\cdot)$.

Consider next functions $h^i(\cdot)$, $i = 1, \ldots, M$, from $\mathbb{R}^n$ into $\mathbb{R}^{d_i}$, which are bounded and Hölder continuous. We shall denote by L the infinitesimal generator of the Markov process $x(\cdot)$

$$L := \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} \tag{2.1}$$

or in divergence form

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x)\frac{\partial}{\partial x_j} - \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} \tag{2.1a}$$

where

$$a_i(x) := -f_i(x) + \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \tag{2.1b}$$

Let us next consider an *impulsive control* defined as follows. There is a sequence $\tau_1 < \tau_2 \ldots < \tau_k < \ldots$ of increasing $\mathcal{F}_t$-stopping times. To each time τ_i we attach an $\mathcal{F}_{\tau_i}$-measurable random variable u_i with values in the set of integers { 1, 2, ..., N } [1]. We define

$$u(t) = u_i, \quad \tau_i \leq t < \tau_{i+1}, \quad i = 0, 1, 2, \ldots \tag{2.2}$$

and set $\tau_0 = 0$. We require that

$$\tau_i \uparrow T \text{ as } i \uparrow \infty, \tag{2.3}$$

while $\tau_k = T$ is possible for some finite k.

Let ν_i be the element of $\mathcal{N}$, corresponding to u_i via (1.25).

Then define

$$h(x, u(t)) := h(x, \nu_i), \qquad \tau_i \leq t < \tau_{i+1}, \tag{2.4}$$

where $h(x, \nu)$ is defined by (1.7), in terms of the given functions $h^i(\cdot)$. Clearly $h(\cdot, u(t))$ maps $\mathbb{R}^n$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ and is obviously bounded and Hölder continuous in x. Define also

$$r(u(t)) := r(\nu_i), \qquad \tau_i \leq t < \tau_{i+1}, \tag{2.5}$$

where $r(\cdot)$ is defined by (1.9), in terms of the given matrices R_i, $i = 1, 2, \ldots, M$. Clearly $r(u(t))$ maps $\mathbb{R}^D$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ but it is *singular*. Next we define $\tilde{h}(x, \nu)$ to be the vector valued function

$$\tilde{h}(x, \nu) := \begin{bmatrix} R_1^{-1/2} h^1(x) \chi_{\{\nu\}}(1) \\ \vdots \\ R_i^{-1/2} h^i(x) \chi_{\{\nu\}}(i) \\ \vdots \\ R_M^{-1/2} h^M(x) \chi_{\{\nu\}}(M) \end{bmatrix} \tag{2.6}$$

with $\chi_{\{\nu\}}(i)$ defined as in (1.8). Let

$$\tilde{h}(x, u(t)) \ := \ \tilde{h}(x, \nu_i), \qquad \tau_i \leq t < \tau_{i+1}. \tag{2.7}$$

Clearly $\tilde{h}(\cdot, u(t))$ maps $\mathbb{R}^n$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ and is obviously bounded and Hölder continuous in x. We shall refer to $u(\cdot)$ as the *impulsive control*. As we shall see, it describes essentially the decision to select at a sequence of decision times one of the functions $h(\cdot, k)$, $k \in \{1, 2, \ldots, N\}$. This is the precise mathematical implementation of the sensor selection decision described in the introduction.

To see that indeed this is the case, we can, with the above preparation, use Girsanov's measure transformation method. Let us then consider the process

$$\varsigma(t) = \exp\{\int_0^t \tilde{h}(x(s), u(s))^T dz(s) - \frac{1}{2}\int_0^t \|\tilde{h}(x(s), u(s))\|^2 ds\} \tag{2.8}$$

[1] Recall that $N = 2^M$ and the binary representation of each integer 1, 2, ..., N determines a sensor activation configuration by (1.25).

where ^{T} denotes transpose, $\|\cdot\|$ is the $\mathbb{R}^D$ norm. Note that the process $u(t)$ is adapted to $\mathcal{F}_t$. Then since $z(\cdot)$ is adapted to $\mathcal{F}_t^w \subset \mathcal{F}_t$ and $u(\cdot)$ is *cadlag* [8], (2.8) is well defined. Moreover since $\bar{h}$ is bounded, by Girsanov's theorem [8], [14], $\varsigma(\cdot)$ is an $\mathcal{F}_t$-martingale. We can thus define a change of probability measure

$$\frac{dP^{u(\cdot)}}{dP}\Big|_{\mathcal{F}_t} = \varsigma(t) \tag{2.9}$$

and consider the process

$$v(t) = z(t) - \int_0^t \bar{h}(x(s), u(s))ds. \tag{2.10}$$

By Girsanov's theorem [8], [14], under the probability measure $P^{u(\cdot)}$ on $(\Omega, \mathcal{A})$, $v(\cdot)$ is a standard $\mathcal{F}_t$-Wiener process with values in $\mathbb{R}^D$. Furthermore, by the independence of $w(\cdot)$ and $z(\cdot)$, $w(\cdot)$ remains a standard $\mathbb{R}^n$-valued, $\mathcal{F}_t$-Wiener process which is independent of $v(\cdot)$. Finally ξ remains independent of $w(\cdot)$, $v(\cdot)$ while keeping its probability law, denoted by π_0. Thus $x(\cdot)$ also retains its probability law under $P^{u(\cdot)}$.

To relate this construction, i.e. (2.2) - (2.10) with the M noisy observations (sensors) loosely described in the introduction (c.f. in particular eq. (1.6)), observe that (2.10) can be written as

$$r(u(t))dz(t) = h(x(t), u(t))dt + r(u(t))dv(t) \tag{2.11}$$

in view of (1.7), (1.9), (2.4), (2.5), (2.6) and (2.7). Indeed

$$\begin{aligned}
r(u(t))\bar{h}(x,(u(t))) &= \begin{bmatrix} R_1^{1/2}\chi_{\{\nu_i\}}(1) & 0 & 0 \\ & & \\ 0 & R_2^{1/2}\chi_{\{\nu_i\}}(2) & 0 \\ & \ddots & \\ 0 & 0 & R_M^{1/2}\chi_{\{\nu_i\}}(M) \end{bmatrix} \begin{bmatrix} R_1^{-1/2}h^1(x)\chi_{\{\nu_i\}}(1) \\ \\ R_2^{-1/2}h^2(x)\chi_{\{\nu_i\}}(2) \\ \vdots \\ R_M^{-1/2}h^M(x)\chi_{\{\nu_i\}}(M) \end{bmatrix} \\
&= h(x,\nu_i), \qquad \tau_i \le t < \tau_{i+1}.
\end{aligned} \tag{2.12}$$

To give a precise meaning to (1.2), or (1.6), let us introduce the process

$$y(t, u(t)) := y^{\nu_i}(t), \qquad \tau_i \le t < \tau_{i+1} \tag{2.13}$$

where

$$dy^{\nu_i}(t) := r(\nu_i)\, dz(t) = h(x(t), \nu_i)\, dt + r(\nu_i)\, dv(t). \tag{2.14}$$

It is clear that if we select $u(t) = \nu$, $\forall t$, where ν has 0 everywhere except for one 1 in the i^{th} location, then (1.2) results. It is also rather plain that $y^\nu(t) \in \mathbb{R}^{D_\nu}$ and that in this case the Wiener process $r(\nu)v(\cdot)$ is also D_ν-dimensional (see (1.16) for the definition of D_ν). The process $y^{\nu_i}(t)$ represents exactly the observation which is available in $[\tau_i, \tau_{i+1})$.

The next issue that we wish to clarify relates to the measurability question that we discussed in section 1.2, after eq. (1.18). For any $u(\cdot)$, given the construction of $y(\cdot, u(\cdot))$, above we can now consider $\mathcal{F}_t^{y(\cdot,u(\cdot))}$ as defined by (1.18). We shall say that $u(\cdot)$ is *admissible*, denoted $u \in U_{ad}$, if $u(t)$ is $\mathcal{F}_t^{y(\cdot,u(\cdot))}$ measurable, $t > 0$, where $\mathcal{F}_t^{y(\cdot,u(\cdot))}$ is constructed as above. This more precisely means that the τ_i are $\mathcal{F}_t^{y(\cdot,u(\cdot))}$-stopping times or that

$$\{\tau_i < t\} \subset \mathcal{F}_t^{y(\cdot,u(\cdot))} \tag{2.15}$$

and that

$$\nu_i \in \mathcal{F}_{\tau_i}^{y(\cdot,u(\cdot))}. \tag{2.16}$$

Note that since $\mathcal{F}_t^{y(\cdot,u(\cdot))} \subset \mathcal{F}_t$ for any sensor schedule $u(\cdot)$ adapted to $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, if τ_i are $\mathcal{F}_t^{y(\cdot,u(\cdot))}$-stopping times they are also $\mathcal{F}_t$-stopping times, and the above construction (2.8) - (2.14) is still valid. The implication of (2.15), (2.16) is that one should check *that an optimizing strategy, obtained by some procedure, must satisfy the admissibility conditions*. Clearly U_{ad} is nonempty as strategies $u(t) = \nu, \ \ t \in [O, T]$, obviously are admissible. Also strategies with fixed switchings are also admissible. Note that for an admissible control $\mathcal{F}_t^{y(\cdot,u(\cdot))} \subset \mathcal{F}_t^{x}$.

We have thus established in this section the precise mathematical models of nonlinear filtering problems where selection of sensors is possible. In particular we have succeeded in circumventing the subtleties associated with the definition of admissible sensor schedules discussed in section 1.2.[2]

3.2.2 The optimization problem

For the dynamical system described in 2.1, we consider now the cost functional (1.19) where the underlying probability measure is $P^{u(\cdot)}$. As indicated in the introduction, the general problem where the function ϕ will be in a nice class, e.g., bounded C^2, or polynomial, or C^∞ can be treated along identical lines. To simplify the notation we have chosen to formulate the problem for $\phi(x) = x$. The technical difficulties for this case are identical to the ones in the more general cases discussed above, particularly since this $\phi(\cdot)$ is unbounded on $\mathbb{R}^n$. For this choice the selection of the optimal estimator $\hat{\phi}(T)$ is the conditional mean

$$\hat{\phi}(T) = E^{u(\cdot)}\{x(T) \mid \mathcal{F}_T^{y(\cdot,u(\cdot))}\}, \tag{2.17}$$

where $E^{u(\cdot)}$ denotes expectation with respect to $P^{u(\cdot)}$. Let $\mu(u,t)$ denote the conditional probability measure of $x(t)$, given $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, on $\mathbb{R}^n$. It is convenient to express (2.17) as a vector valued functional of $\mu(u,t)$

$$\hat{\phi}(T) = \Phi(\mu(u,T)) = \int_{\mathbb{R}^n} x\,d\mu(u,T). \tag{2.18}$$

We shall further assume that the running and switching cost functions $c_i(\cdot), k_{ij}(\cdot)$, $i, j \in \{1, \ldots, N\}$, introduced in (1.4) and (1.5) have the following regularity

$$c_i(\cdot), k_{ij}(\cdot) \text{ are in } C_b(\mathbb{R}^n) \text{ (i. e. bounded and continuous)} \tag{2.19}$$

[2]Since $r(u(t))$ is a singular matrix, this stage is more delicate than in standard stochastic control theory, where $\mathcal{F}_t^{x}$ would suffice.

As a result of this simple transformation we can rewrite the cost as a function of the impulsive control $u(\cdot)$ only (i.e. the selection of $\hat{\phi}(\cdot)$ has been eliminated):

$$\begin{aligned} J(u(\cdot)) &= E^{u(\cdot)}\{\|x(T) - \Phi(\mu(u,T))\|^2 + \int_0^T c(x(t), u(t))dt \\ &\quad + \sum_{j=1}^{\infty} k(x(\tau_j), u(\tau_{j-1}), u(\tau_j))\chi_{\tau_j<T}\}, \end{aligned} \tag{2.20}$$

where $\chi_{\tau_i<T}$ is the characteristic function of the Ω-set $\{\omega;\ \tau_i(\omega) < T\}$. We further assume that the switching costs are uniformly bounded below

$$k(x,i,j) \geq k_o,\ \ x \in \mathbb{R}^n,\ i,\ j \in \{1,\ldots,N\} \tag{2.21}$$

with k_0 a positive constant. Note that as a consequence of (2.20) if for some admissible $u(\cdot)$ with positive probability, the number of times $\tau_i < T$ is infinite, then the cost $J(u(\cdot))$ will be infinite. Therefore for T finite the optimal policy will exhibit a finite number of sensor switchings.

The optimal sensor selection problem can now be stated precisely as the optimization problem

$\mathcal{P}$: *Find an admissible impulsive control* $u^*(\cdot)$ *such that*

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in U_{ad}} J(u(\cdot)), \tag{2.22}$$

where U_{ad} are all impulsive control strategies adapted to $\mathcal{F}^{y(\cdot,u(\cdot))}$, or equivalently satisfying (2.15), (2.16). Problem $\mathcal{P}$ is a *non-standard* stochastic control problem of a partially observed diffusion.

3.2.3 The equivalent fully observed problem

In this section we transform the problem of section 2.2, to a fully observed stochastic control problem, by introducing appropriate Zakai equations. As is customary in the theory of nonlinear filtering [1], [2], [3], [4], let us introduce the operator

$$p(u(\cdot),t)(\psi) = E\{\varsigma(t)\psi(x(t)) \mid \mathcal{F}_t^{y(\cdot,u(\cdot))}\} \tag{2.23}$$

for each impulsive control $u(\cdot)$. The notation is chosen so as to emphasize the dependence on $u(\cdot)$, which is due to the dependence of $\varsigma(\cdot)$ on $u(\cdot)$ as introduced in eq. (2.8).[3] The operator (2.23) maps the set of Borel bounded functions on $\mathbb{R}^n$, into the set of real valued stochastic processes adapted to $\mathcal{F}_t^{y(\cdot,u(\cdot))}$. Note that $p(u(\cdot),t)$ can be viewed as a positive finite measure on $\mathbb{R}^n$. It is the *unnormalized conditional probability measure* of $x(t)$ given $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, [1], [2].

[3]But the expectation is with respect to P and not $P^{u(\cdot)}$.

With the help of these measures we can rewrite the various cost terms in (2.20) as follows:

$$\begin{aligned} E^{u(\cdot)}\{\|x(T)-\Phi(\mu(u,T))\|^2\} &= E\{\varsigma(T)\|x(T)-\Phi(\mu(u,T))\|^2\} \\ &= E\{p(u(\cdot),T)(\theta)\}, \end{aligned} \tag{2.24}$$

where

$$\theta(x) := \|x - \frac{p(u(\cdot),T)(\chi)}{p(u(\cdot),T)(\mathbb{1})}\|^2, \tag{2.25}$$

with χ representing the function $\chi(x) := x$ and $\mathbb{1}$ the function $\mathbb{1}(x) := 1,\ x \in \mathbb{R}^n$. A straightforward computation implies that

$$E^{u(\cdot)}\{\|x(T)-\Phi(\mu(u,T))\|^2\} = E\{\Psi(p(u(\cdot),T))\} \tag{2.26}$$

where Ψ is the functional on finite measures on $\mathbb{R}^n$ defined by

$$\Psi(\mu) = \mu(\chi^2) - \frac{\|\mu(\chi)\|^2}{\mu(\mathbb{1})} \tag{2.27}$$

where $\chi^2(x) = \|x\|^2$, $x \in \mathbb{R}^n$, and μ is any finite measure on $\mathbb{R}^n$ such that the quantities $\mu(\chi^2)$ and $\mu(\chi)$ make sense.

Next

$$\begin{aligned} E^{u(\cdot)}\{\int_0^T c(x(t),u(t))dt\} &= E\{\varsigma(T)\int_0^T c(x(t),u(t))dt\} \\ &= E\{\int_0^T E\{\varsigma(T)c(x(t),u(t))\,|\mathcal{F}_t\}dt\} \\ &= E\{\int_0^T E\{\varsigma(T)\,|\mathcal{F}_t\}c(x(t),u(t))dt\} \\ &= E\{\int_0^T \varsigma(t)c(x(t),u(t))dt\}, \end{aligned} \tag{2.28}$$

because $x(t), u(t)$ are measurable with respect to $\mathcal{F}_t$ and $\varsigma(\cdot)$ is an $\mathcal{F}_t$-martingale. Now define a map C with values in $C_b(\mathbb{R}^n)$ via

$$C(u_i) := c_{u_i}(\cdot),\ u_i \in \{1,2,\ldots,N\}. \tag{2.29}$$

Then in view of (2.29), (2.23), we can rewrite (2.28) as

$$\begin{aligned} E^{u(\cdot)}\{\int_0^T c(x(t),u(t))dt\} &= E\{\int_0^T E\{\varsigma(t)c(x(t),u(t))|\mathcal{F}_t^{y(\cdot,u(\cdot))}\}dt\} \\ &= E\{\int_0^T p(u(\cdot),t)(C(u(t))dt\}. \end{aligned} \tag{2.30}$$

Finally

$$\begin{aligned} E^{u(\cdot)}\{k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\} &= E\{\varsigma(\tau_i)k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\} \\ &= E\{E\{\varsigma(\tau_i)k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\,|\mathcal{F}_{\tau_i}^{y(\cdot,u(\cdot))}\}\} \\ &= E\{p(u(\cdot),\tau_i)(K(u(\tau_{i-1}),u(\tau_i)))\chi_{\tau_i<T}\}. \end{aligned} \tag{2.31}$$

Here we have introduced the function K with values in $C_b(\mathbb{R}^n)$, via

$$K(u_i, u_j) = \mathbf{k}_{u_i,u_j}(\cdot), \quad u_i,\ u_j \in \{1,2,\ldots,N\}, \tag{2.32}$$

and we utilized the admissibility of $u(\cdot)$. Note that in the simpler case where $\mathbf{c}_i(\cdot), \mathbf{k}_{ij}(\cdot)$, $i,\ j \in \{1,2,\ldots,N\}$ are constant independent of x, (2.30) simplifies to

$$E^{u(\cdot)}\{\int_0^T c(x(t),u(t))dt\} = E\{\int_0^T p(u(\cdot),t)(\mathbb{1})\mathbf{c}_{u(t)}dt\} \tag{2.33}$$

and (2.31) simplifies to

$$E^{u(\cdot)}\{k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\} = E\{\mathbf{k}_{u_{i-1},u_i}\chi_{\tau_i<T}p(u(\cdot),\tau_i)(\mathbb{1})\}. \tag{2.34}$$

Utilizing (2.26), (2.30), (2.31) we can rewrite the cost corresponding to policy $u(\cdot)$, given in (2.20), as follows

$$\begin{aligned} J(u(\cdot)) &= E\{\Psi(p(u(\cdot),T)) + \int_0^T p(u(\cdot),t)(C(u(t)))dt \\ &+ \sum_{i=1}^{\infty} p(u(\cdot),\tau_i)(K(u_{i-1},u_i))\chi_{\tau_i<T}\}. \end{aligned} \tag{2.35}$$

In (2.35) we have succeeded in displaying the cost as a functional of the unnormalized conditional measure $p(u(\cdot),\cdot)$ which is the "information" state of the equivalent fully observed stochastic control problem. To complete this transformation we need to derive the evolution equation for $p(u(\cdot),\cdot)$, i.e. the Zakai equation. We turn into this problem next and derive a weak form of the Zakai equation for $p(u(\cdot),\cdot)$ in the following lemma. Here $C_b^{2,1}$ denotes the space of all functions $\psi(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$ which are bounded, continuous together with their first and second derivatives with respect to x, and first derivatives with respect to t.

Lemma 2.1: *For any $\psi \in C_b^{2,1}$ we have the relation*

$$\begin{aligned} p(u(\cdot),t)(\tilde{\psi}(t)) &= \pi_0(\tilde{\psi}(0)) + \int_0^t p(u(\cdot),s)(\frac{\partial\tilde{\psi}}{\partial s} + L\tilde{\psi})ds \\ &+ \int_0^t \sum_{i=1}^{D} p(u(\cdot),s)(\tilde{H}_i(u(s))\tilde{\psi}(s))dz_i(s) \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} \left[\tilde{H}_i(u(s))\phi\right](x) &:= \tilde{h}_i(x,u(s))\phi(x), \ \ i = 1,2,\ldots,D, \ \ \phi \in C_b^2 \\ \tilde{\psi}(s)(x) &:= \psi(x,s), \end{aligned} \tag{2.37}$$

and $\tilde{h}_i$ is the i^{th} component of $\tilde{h}$ (see (2.6)).

Proof:

Let $\beta(\cdot) \in L^\infty(0,T;\mathbb{R}^D)$ given and consider the $\mathcal{F}_t$-martingale $\rho(t)$, defined by

$$d\rho(t) = \rho(t)\beta(t)^T dz(t), \quad \rho(0) = 1. \tag{2.38}$$

Recall that by definition of $\varsigma(t)$ (c.f. eq. (2.8))

$$d\varsigma(t) = \varsigma(t)\tilde{h}(x(t),u(t))^T dz(t), \qquad \varsigma(0) = 1. \tag{2.39}$$

Therefore by Itô's rule [8]

$$\begin{aligned} d(\varsigma(t)\rho(t)) &= \varsigma(t)\rho(t)[(\tilde{h}(x(t),u(t)) + \beta(t))^T dz(t) \\ &+ \tilde{h}^T(x(t),u(t))\beta(t)dt] \\ \varsigma(0)\rho(0) &= 1, \end{aligned} \tag{2.40}$$

and since $\psi \in C_b^{2,1}$

$$\begin{aligned} d\psi(x(t),t) &= (\frac{\partial\psi(x(t),t)}{\partial t} + L\psi(x(t),t))dt \\ &+ [\nabla\psi(x(t),t)]^T g(x(t))dw(t), \end{aligned} \tag{2.41}$$

where L is given in (2.1). Therefore suppressing some arguments for ease of notation

$$\begin{aligned} d[\psi(x(t),t)\ \varsigma(t)\rho(t)] &= \varsigma(t)\rho(t)[(\frac{\partial\psi}{\partial t} + L\psi + \tilde{h}^T\beta\psi)dt \\ &+ \nabla\psi^T g dw(t) + \psi(\tilde{h}+\beta)^T dz(t)]. \end{aligned} \tag{2.42}$$

In (2.41), (2.42) we used the notation $\nabla\psi = (\frac{\partial\psi}{\partial x_1},\ldots,\frac{\partial\psi}{\partial x_n})^T$. Integrating (2.42), and taking expectations we deduce

$$E\{\psi(x(t),t)\varsigma(t)\rho(t)\} = \pi_0(\tilde{\psi}(0)) + E\{\int_0^t \varsigma(s)\rho(s)[\frac{\partial\psi}{\partial s} + L\psi + \tilde{h}^T\beta\psi]ds\}. \tag{2.43}$$

We can then write

$$\begin{aligned} E\{\int_0^t \varsigma(s)\rho(s)[\frac{\partial\psi}{\partial s} + L\psi]ds\} &= E\{\int_0^t E\{\rho(s)\varsigma(s)(\frac{\partial\psi}{\partial s} + L\psi) \mid \mathcal{F}_s^{y(\cdot,u(\cdot))}\}ds\} \\ &= E\{\int_0^t \rho(s)p(u(\cdot),s)(\frac{\partial\psi}{\partial s} + L\psi)ds\} \\ &= E\{\rho(t)\int_0^t p(u(\cdot),s)(\frac{\partial\psi}{\partial s} + L\psi)ds\} \end{aligned} \tag{2.44}$$

by virtue of the $\mathcal{F}_t$-martingale property of $\rho(\cdot)$. Similarly

$$\begin{aligned} & E\{\int_0^t \varsigma(s)\rho(s)\tilde{h}(x(s),u(s))^T\beta(s)\psi(x(s),s)ds\} \\ &= E\{\rho(t)\int_0^t \varsigma(s)\psi(x(s),s)\tilde{h}(x(s),u(s))^T dz(s)\} \\ &= E\{\rho(t)\int_0^t \sum_{i=1}^{D} p(u(\cdot),s)(\tilde{h}_i(\cdot,u(s))\psi(\cdot,s))dz_i(s)\}, \end{aligned} \tag{2.45}$$

where in the first equality we have used the representation $\rho(t) = 1 + \int_0^t \rho(s)\beta(s)^T dz(s)$, and the well known insomorphism between Itô stochastic integrals and L^2 [8]. Finally

$$E\{\psi(x(t),t)\varsigma(t)\rho(t)\} = E\{\rho(t)p(u(\cdot),t)(\tilde{\psi}(t))\}. \tag{2.46}$$

Using (2.44), (2.45), (2.46) in (2.43) we obtain

$$E\{\rho(t)[p(u(\cdot),t)(\tilde{\psi}(t)) - \pi_0(\tilde{\psi}(0)) - \int_0^t p(u(\cdot),s)(\frac{\partial\psi}{\partial s} + L\psi)ds - \int_0^t \sum_{i=1}^{D} p(u(\cdot),s)(\tilde{H}_i(u(s))\tilde{\psi}(s))dz_i(s)]\} = 0. \tag{2.47}$$

We can replace in (2.47) $\rho(t)$ by a linear combination of such variables, with different β. The set of corresponding variables is dense in $L^2(\Omega, \mathcal{F}_t^z, P)$. However, the random variable in the brackets in the right hand side of (2.47) is clearly in $L^2(\Omega, \mathcal{F}_t^{y(\cdot,u(\cdot))}, P)$ and therefore in $L^2(\Omega, \mathcal{F}_t^z, P)$ since $\mathcal{F}_t^{y(\cdot,u(\cdot))} \subset \mathcal{F}_t^z$. Then (2.47) implies the result of the lemma (2.36).

As a remark we would like to note that the assumed nondegeneracy of $x(\cdot)$, implies that the solution of (2.36) is unique. This can be proved in general under our working hypotheses, for solutions which are measure-valued processes. Here we outline such a proof for the case when these conditional measures are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$; i.e., in the case unnormalized conditional densities exist. For this we need to assume in addition that

$$\pi_0 \textit{ has a density } p_0 \textit{ with respect to Lebesgue measure; } p_0 \in L^2(\mathbb{R}^n) \tag{2.48}$$

Let us denote by L^* the formal adjoint of L (see (2.1), (2.1a), (2.1b)):

$$L^* = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i, \tag{2.49}$$

and consider the Hilbert space form of the Zakai equation [10]

$$\begin{aligned} dp &= L^* p dt + p\tilde{h}(\cdot, u(t))^T dz(t) \\ p(0) &= p_0. \end{aligned} \tag{2.50}$$

The function space in which the solution is sought is

$$L^2(\Omega, \mathcal{A}, P; C(0,T; L^2(\mathbb{R}^n))) \cap L^2_{\mathcal{F}^{y(\cdot,u(\cdot))}}(0,T; H^1(\mathbb{R}^n)) \tag{2.51}$$

Here H^1 is the usual Sobolev space on $\mathbb{R}^n$ [11] and the subindex $\mathcal{F}^{y(\cdot,u(\cdot))}$ in the second L^2 space, denotes that the solution is adapted to the filtration $\mathcal{F}_t^{y(\cdot),u(\cdot))}$, $t \geq 0$. It follows from the results of E. Pardoux [11], that there exists a unique solution of (2.49) in the function space (2.50), under the assumptions made here. We can then establish the following.

Lemma 2.2: The following property holds

$$p(u(\cdot),t)(\psi) = (p(u(\cdot),t), \psi), \tag{2.52}$$

$\forall \psi$ in $L^2(\mathbb{R}^n)$ and bounded, where $(\cdot,\cdot)$ denotes inner product in $L^2(\mathbb{R}^n)$.

Proof:

By slight abuse of notation we use the same symbol to denote the conditional unormalized measure *and* density (whenever the latter exists). Let us prove inductively that

$$p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1}))(\psi) = (p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1})),\psi), \tag{2.53}$$

where the left hand side notation refers to the measure appearing in (2.36), while the right hand side notation to the solution of (2.50), which is uniquely defined. Suppose then that (2.53) holds for i-1, and therefore in particular

$$p(u(\cdot),\tau_i)(\psi) = (p(u(\cdot),\tau_i),\psi), \forall\psi. \tag{2.54}$$

Consider now the solution η of

$$\begin{aligned} \frac{\partial \eta}{\partial s} + L\eta &= -\eta \tilde{h}(\cdot,u(s))^T \beta(s), \quad s \in (\tau_i, \tau_i \vee (t \wedge \tau_{i+1})) \\ \eta(x,\tau_i \vee (t \wedge \tau_{i+1})) &= \psi(x) \end{aligned} \tag{2.55}$$

where $\psi \in C_0^\infty(I\!R^n)$ and β is a smooth deterministic function with values in $I\!R^D$. From the assumptions on f, g and h^i (it is here that we use the assumed Hölder continuity of h^i), we can assert that the solution of (2.55) belongs to $C_b^{2,1}(I\!R^n \times (\tau_i, \tau_i \vee (t \wedge \tau_{i+1})))$, for any sample ω, [11]. Therefore (2.36) implies (using (2.55))

$$\begin{aligned} p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1}))(\psi) &= p(u(\cdot),\tau_i)(\bar{\eta}(\tau_i)) \\ &- \int_{\tau_i}^{\tau_i \vee (t\wedge\tau_{i+1})} \sum_{j=1}^D p(u(\cdot),s)(\tilde{H}_j(u(s))\bar{\eta}(s))\beta_j(s)ds \\ &+ \int_{\tau_i}^{\tau_i \vee (t\wedge\tau_{i+1})} \sum_{j=1}^D p(u(\cdot),s)(\tilde{H}_j(u(s))\bar{\eta}(s))dz_j(s), \end{aligned} \tag{2.56}$$

where $\tilde{H}_j$ is as defined in lemma 2.1, and $\bar{\eta}(s)(x) := \eta(x,s)$. Therefore by Itô's rule

$$\begin{aligned} p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1}))(\psi)\rho(\tau_i \vee (t \wedge \tau_{i+1})) &= p(u(\cdot),\tau_i)(\bar{\eta}(\tau_i))\rho(\tau_i) \\ &+ \int_{\tau_i}^{\tau_i \vee (t\wedge\tau_{i+1})} [\rho(s) \sum_{j=1}^D p(u(\cdot),s)(\tilde{H}_j(u(s))\bar{\eta}(s)) \\ &+ \int_{\tau_i}^{\tau_i \vee (t\wedge\tau_{i+1})} \rho(s) \sum_{j=1}^D p(u(\cdot),s)(\bar{\eta}(s))\beta_j(s)]dz_j(s). \end{aligned} \tag{2.57}$$

Hence

$$\begin{aligned} &E\{p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1}))(\psi)\rho(\tau_i \vee (t \wedge \tau_{i+1}))\} \\ &= E\{p(u(\cdot),\tau_i)(\bar{\eta}(\tau_i))\rho(\tau_i)\}. \end{aligned} \tag{2.58}$$

On the other hand from (2.50) and (2.55) we obtain

$$\begin{aligned} (p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1})),\psi) &= (p(u(\cdot),\tau_i),\bar{\eta}(\tau_i)) \\ &+ \int_{\tau_i}^{\tau_i \vee (t\wedge\tau_{i+1})} \sum_{j=1}^D (p(u(\cdot),s)\tilde{h}_j(\cdot,u(s)),\bar{\eta}(s))dz_j(s) \\ &- \int_{\tau_i}^{\tau_i \vee (t\wedge\tau_{i+1})} \sum_{j=1}^D (p(u(\cdot),s),\tilde{H}_j(u(s))\bar{\eta}(s)\beta_j(s)ds, \end{aligned} \tag{2.59}$$

and thus also

$$E\{(p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1})),\psi)\rho(\tau_i \vee (t \wedge \tau_{i+1}\} = E\{(p(u(\cdot),\tau_i),\tilde{\eta}(\tau_i))\rho(\tau_i\}. \tag{2.60}$$

But from the inductive hypothesis (2.54), the right hand sides of (2.58) and (2.60) are equal. Hence the left hand sides coincide. Varying β, we easily deduce that (2.53) holds, at least for $\psi \in C_0^\infty(I\!R^n)$, which is sufficient to conclude the proof of the lemma.

With this result we can rewrite the cost (2.35) as follows

$$\begin{aligned} J(u(\cdot)) &= E\{\Psi(p(u(\cdot),T)) + \int_0^T (p(u(\cdot),t), C(u(t)))dt \\ &+ \sum_{i=1}^{\infty} \chi_{\tau_i<T}(p(u(\cdot),\tau_i), K(u_{i-1},u_i))\} \end{aligned} \tag{2.61}$$

where (see (2.27))

$$\Psi(p(u(\cdot),T)) = (p(u(\cdot),T),\chi^2) - \frac{\|(p(u(\cdot),T),\chi)\|^2}{(p(u(\cdot),T),\mathbb{1})}. \tag{2.62}$$

Since the expression (2.62) involves unbounded functions we have to show that it makes sense.

At this point it is useful to introduce a weighted Hilbert space in order to express $\Psi(p(u(\cdot),T))$ in a more convenient form. To this end let

$$\mu(x) = 1 + \| x \|^4 \tag{2.63}$$

and $L^2(I\!R^n;\mu)$ denotes the space of functions φ such that $\varphi\mu \in L^2(I\!R^n)$. Define in a similar way the space $L^1(I\!R^n;\mu)$. From the discussion of existence and uniqueness of solutions of (2.50) in the functional space (2.51) and if

$$p_0 \in L^2(I\!R^n;\mu) \cap L^1(I\!R^n;\mu),$$

it is easy to check that (2.50), under the assumptions made in section 2.1, has a unique solution in the space

$$L^2(\Omega, \mathcal{A}, P; C(0,T; L^2(I\!R^n;\mu) \cap L^1(I\!R^n;\mu))) \cap L^2(0,T; H^1(I\!R^n;\mu)) \tag{2.64}$$

where $H^1(I\!R^n;\mu)$ is the obvious modification of $H^1(I\!R^n)$. This justifies that the quantities arising in (2.62) have a meaning.

We note that $J(u(\cdot))$ is indexed implicitly (we do not include this in our notation) by π_0 (or p_0) and $u(0) = j$, $j \in \{1,\ldots,N\}$ which is determistic since it is $\mathcal{F}_0^z$-measurable, by construction.

We close this section by rewriting the dynamics (2.50), in terms of the originally given observation nonlinearities h^i, and with forcing inputs the processes $y^i(\cdot)$ introduced in (2.13), (2.14). In view of (2.5), (2.6), (2.7), (2.13), (2.14) we have

$$\tilde{h}(\cdot,u(t))^T dz(t) = \sum_{j=1}^{M} h^{j^T}(\cdot)\chi_{\{\nu_i\}}(j)R_j^{-1/2}dz_j(t), \tau_i \le t < \tau_{i+1}$$

(where we have written $z = [z_1, z_2, \ldots z_M]^T$)

$$
\begin{aligned}
&= \sum_{j=1}^{M} h^{j^T}(\cdot)\chi_{\{\nu_i\}}(j) R_j^{-1} R_j^{1/2} \chi_{\{\nu_i\}}(j) dz_j(t), \ \tau_i \leq t < \tau_{i+1} \\
&= \delta(\cdot, \nu_i)^T dy(t, \nu_i), \ \tau_i \leq t < \tau_{i+1} \\
&=: \delta(\cdot, u(t))^T dy(t, u(t)),
\end{aligned}
$$

where

$$
\delta(x, \nu) = \begin{bmatrix} R_1^{-1} h^1(x)\chi_{\{\nu\}}(1) \\ \vdots \\ R_i^{-1} h^i(x)\chi_{\{\nu\}}(i) \\ \vdots \\ R_M^{-1} h^M(x)\chi_{\{\nu\}}(M) \end{bmatrix} \tag{2.65}
$$

Therefore the system dynamics (2.50) can be written equivalently

$$
\begin{aligned}
dp(u(\cdot), t) &= L^* p(u(\cdot), t)dt + p(u(\cdot), t)\delta(\cdot, u(t))^T dy(t, u(\cdot)) \\
p(u(\cdot), 0) &= p_0,
\end{aligned} \tag{2.66}
$$

where $y(t, u(t))$ is defined in (2.13), (2.14). This makes precise the construction of a Zakai equation driven by "controlled" observations alluded to in the introduction. It also becomes now clear that the spaces described by (2.51), (2.64) are the appropriate ones as far as solutions of (2.50) or (2.66) are concerned.

3.3 The Solution of the Optimization Problem

3.3.1 Setting up a system of quasi–variational inequalities

Let us consider the Banach space $H = L^2(\mathbb{R}^n; \mu) \cap L^1(\mathbb{R}^n; \mu)$ and the metric space H^+ of positive elements of H. Let

$$
\begin{aligned}
\mathcal{B} &:= \text{space of Borel measurable, bounded functions on} H^+ \\
\mathcal{C} &:= \text{space of uniformly continuous, bounded functions on} H^+.
\end{aligned} \tag{3.1}
$$

Let us now define semigroups $\Phi_j(t)$ on $\mathcal{B}$ or $\mathcal{C}$ as follows. Consider (2.50) with fixed schedule $u(t) = j$, and let p_j denote the corresponding density $p(\cdot, j)$. Then for $j \in \{1, 2, \ldots, N\}$

$$
dp_j = L^* p_j \, dt + p_j \tilde{h}^{j^T} dz(t), \ p_j(0) = \pi, \tag{3.2}
$$

where

$$
\tilde{h}^j := \tilde{h}(\cdot, j). \tag{3.3}
$$

We set

$$
\Phi_j(t)(F)(\pi) = E\{F(p_{j,\pi}(t))\}, \ \ F \in \mathcal{B} \ \text{ or } \ \mathcal{C}, \tag{3.4}
$$

where $p_{j,\pi}$ indicates the solution of (3.2) with initial value π. It is easy to see that Φ_j is a semigroup since $p_j(t)$ is a Markov process with values in H^+. It is also useful to introduce the subspaces $\mathcal{B}_1$ and $\mathcal{C}_1$ of functions such that

$$\|F\|_1 \sup_{\pi\in H^+} \frac{|F(\pi)|}{1+\|\pi\|_\mu} < \infty \tag{3.5}$$

where $\| \pi \|_\mu = \| \pi \|_{L^1(R^n,\mu)}$. The spaces $\mathcal{B}_1$ and $\mathcal{C}_1$ are also Banach spaces. They are needed, because we shall encounter functionals with linear growth in the cost function (2.61). To simplify the statement and analysis of the quasi-variational inequalities that solve the optimization problem considered here, we give the details for the case N =2 only in the sequel. We shall insert remarks to indicate how the results should be modified for the general case. Let us introduce the notation

$$\begin{aligned} C_i &:= C(i,\cdot), \quad i=1,2, \\ K_1 &:= K(1,2) \\ K_2 &:= K(2,1). \end{aligned} \tag{3.6}$$

Since C_1, C_2, K_1, K_2 are bounded functions, one can utilize them to define elements of $\mathcal{C}_1$ via (for example)

$$C_1(\pi) = (C_1, \pi) \tag{3.7}$$

where a slight abuse of notation, in denoting the functional and the function by the same symbol, has been allowed. Similarly the functional on H^+

$$\Psi(\pi) = (\pi, \chi^2) - \frac{\|(\pi,\chi)\|^2}{(\pi,\mathbb{1})} \tag{3.8}$$

belongs to $\mathcal{C}_1$ since it is positive and

$$\Psi(\pi) \le (\pi,\chi^2) \le \|\pi\|_\mu. \tag{3.9}$$

Consider now the set of functionals $U_1(\pi,t)$, $U_2(\pi,t)$ such that

$$\begin{aligned} U_1, U_2 &\in C(0,T;\mathcal{C}_1) \\ U_1(\cdot,t) \ge 0, &\quad U_2(\cdot,t) \ge 0 \\ U_1(\pi,T) &= U_2(\pi,T) = \Psi(\pi) \\ U_1(\pi,t) &\le \Phi_1(s-t)U_1(\pi,s) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda \\ U_2(\pi,t) &\le \Phi_2(s-t)U_2(\pi,s) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \\ \forall s &\ge t \\ U_1(\pi,t) &\le K_1(\pi) + U_2(\pi,t) \\ U_2(\pi,t) &\le K_2(\pi) + U_1(\pi,t). \end{aligned} \tag{3.10}$$

In the sequel we will occasionally use the notation $U_i(s)(\pi) = U_i(\pi,s)$, $\quad i=1,2$.

3.3.2 Existence of a maximum element

We shall refer to (3.10) as the system of quasi-variational inequalities (QVI). Our first objective is to prove the following.

Theorem 3.1. *We assume that the conditions on the data f, g, h^i introduced in section 2.1 hold. Then the set of functionals U_1, U_2 satisfying (3.10) is non-empty and has a maximum element, in the sense that if $\tilde{U}_1, \tilde{U}_2$ denotes this maximum element and U_1, U_2 satisfies (3.10), then*

$\tilde{U}_1 \geq U_1, \tilde{U}_2 \geq U_2$.

The proof will be carried out in several steps. In fact there is some difficulty due to the functional $\Psi(\pi)$. We shall modify it in order to assume that

$$0 \leq \Psi(\pi) \leq \overline{\Psi}(\pi, \mathbb{1}) \tag{3.11}$$

where $\overline{\Psi}$ is a constant. We shall prove the theorem with the additional assumption (3.11), prove the probabilistic interpretation, i.e. the connection with the infimum of (2.61). The probabilistic formula will be next used in an approximation procedure. We can approximate for instance the functional Ψ defined by (3.8) in the following way. Set

$$\Psi_n(\pi) = \int \frac{\pi \|x\|^2}{1 + \frac{\|x\|^2}{n}} dx - \frac{\| \int \frac{\pi x}{(1+\frac{\|x\|^2}{n})^{1/2}} dx \|^2}{\int \pi dx} \tag{3.12}$$

which clearly satisfies (3.11) with $\overline{\Psi} = n$.

Proof of Theorem 3.1 under the assumption (3.11). The set of functionals satisfying (3.10) is a subset of B_1 or C_1 defined in (3.5). However for this subset the norm (3.5) is unnecessarily restrictive. For those functionals it is sufficient to set

$$\begin{aligned} \tilde{H} &= L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \\ \tilde{H}^+ &= \text{set of positive elements of } \tilde{H} \end{aligned} \tag{3.13}$$

and to consider $\tilde{B}_1, \tilde{C}_1$ the space of Borel or continuous functionals on $\tilde{H}^+$ such that

$$\|F\|_1 = \sup_{\pi \in \tilde{H}^+} \frac{|F(\pi)|}{1 + (\pi, \mathbb{1})} < \infty \tag{3.14}$$

We shall then study the system (3.10) with C_1 replaced by $\tilde{C}_1$. Let us notice that

$$H^+ \subset \tilde{H}^+$$

and if one considers a functional F in $\tilde{B}_1$ or $\tilde{C}_1$, its restriction to H^+ belongs to B_1 or C_1; the injection

$$F \to \text{restriction of } F \text{ to } H^+$$

is continuous from $\tilde{B}_1$ or $\tilde{C}_1$ to B_1 or C_1. Therefore replacing in (3.10) C_1 by $\tilde{C}_1$ gives a stronger result.

We shall in the proof omit to write the symbol $\sim$ and write B_1, C_1 instead of $\tilde{B}_1, \tilde{C}_1, H^+$ instead of $\tilde{H}^+$, the norm $\| \; \|_1$ is then given by (3.14).

The proof is then an adaptation of the methods of Bensoussan-Lions [9] to the present case in order to take into account the fact that we use C_1 instead of C.

First note that

$$\|\Phi_1(t)\|_{\mathcal{L}(C_1;C_1)} \leq 1 \tag{3.15}$$

where $\mathcal{L}(C_1; C_1)$ is the space of linear continuous operators from C_1 into itself. Indeed we have

$$\begin{aligned}
\frac{|\Phi_1(t)(F)(\pi)|}{1+(\pi,\mathbb{1})} &= \frac{|E\{\bar{F}(p_{1,\pi}(t))\}|}{1+(\pi,\mathbb{1})} \\
&\leq \|F\|_1 \frac{(1+E(p_{1,\pi}(t),\mathbb{1}))}{1+(\pi,\mathbb{1})} \\
&= \|F\|_1
\end{aligned}$$

since from (3.2)

$$E(p_{1,\pi}(t),\mathbb{1}) = (\pi,\mathbb{1}) \tag{3.16}$$

Therefore

$$\|\Phi_1(t)(F)\|_1 \leq \|F\|_1 \tag{3.17}$$

which implies (3.15).

Note also that a solution of (3.10) will satisfy

$$U_1(\pi,t) \leq \Phi_1(T-t)U_1(\pi,T) + \int_t^T \Phi_1(\lambda-t)C_1(\pi)d\lambda \tag{3.18}$$

and due to positivity, we also have

$$\|U_1(t)\|_1 \leq \|U_1(T)\|_1 + \|C_1\|_1(T-t) \leq \overline{\Psi} + \|C_1\|(T-t) \tag{3.19}$$

where $\|C_1\| = \sup_x C_1(x)$.

As it is customary in the study of QVI we begin with the corresponding obstacle problem,

$$\begin{aligned}
U_1, U_2 &\in C(0,T;C_1) \\
U_1(\cdot,t) &\geq 0, U_2(\cdot,t) \geq 0 \\
U_1(\pi,T) &= U_2(\pi,T) = \Psi(\pi) \\
U_1(\pi,t) &\leq \Phi_1(s-t)U_1(\pi,s) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda \\
U_2(\pi,t) &\leq \Phi_2(s-t)U_2(\pi,s) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \\
\forall s &\geq t \\
U_1(\pi,t) &\leq K_1(\pi) + \varsigma_2(\pi,t) \\
U_2(\pi,t) &\leq K_2(\pi) + \varsigma_1(\pi,t)
\end{aligned} \tag{3.20}$$

where we assume that

$$\begin{aligned}
\varsigma_1, \varsigma_2 &\in C(0,T;C_1) \\
\varsigma_1(\pi,t) \geq 0, &\quad \varsigma_2(\pi,t) \geq 0 \\
\varsigma_1(\pi,T), \varsigma_2(\pi,T) &\geq \Psi(\pi).
\end{aligned} \tag{3.21}$$

We then have the following.

Proposition 3.1: *For ς_1, ς_2 as in* (3.21) *the set of U_1, U_2 satisfying* (3.20) *is not empty and has a maximum element.*

It is clear that for ς_1, ς_2 given, the system of inequalities (3.20) can be decoupled and U_1, U_2 can be considered separately. Let us then omit indices momentarily and consider

$$\begin{aligned}
U &\in C(0,T;C_1) \\
U(\cdot,t) &\geq 0 \\
U(\pi,T) &= \Psi(\pi) \\
U(\pi,t) &\leq \Phi(s-t)U(\pi,s) + \int_t^s \Phi(\lambda - t)C(\pi)d\lambda \\
\forall s &\geq t \\
U(\pi,t) &\leq \varsigma(t)
\end{aligned} \tag{3.22}$$

where ς stands for instance, for $K_1(\pi) + \varsigma_2(\pi,t)$. To prove proposition 3.1, it suffices to show that (3.22) has a maximum element. This can be done by the penalty method. So we look for U_ϵ solving

$$\begin{aligned}
U_\epsilon(t) &= \Phi(s-t)U_\epsilon(s) + \int_t^s \Phi(\lambda-t)[C(\pi) - \frac{1}{\epsilon}(U_\epsilon(\lambda) - \varsigma(\lambda))^+]d\lambda \\
\text{for} \quad & t \leq s \leq T \\
U_\epsilon(T)(\pi) &= \Psi(\pi) \\
U_\epsilon &\in C(0,T;C_1) \\
U_\epsilon(\cdot,t) &\geq 0.
\end{aligned} \tag{3.23}$$

We can then assert

Lemma 3.1 *There is a unique solution of* (3.23).

Proof: Notice that (3.23) is equivalent to

$$U_\epsilon(t) = \Phi(T-t)U_\epsilon(T) + \int_t^T \Phi(\lambda-t)[C(\pi) - \frac{1}{\epsilon}(U_\epsilon(\lambda) - \varsigma(\lambda))^+]d\lambda \tag{3.24}$$

and also to

$$\begin{aligned}
U_\epsilon(t) \quad = \quad & e^{-\frac{1}{\epsilon}(T-t)}\Phi(T-t)\Psi(\pi) + \int_t^T e^{-\frac{1}{\epsilon}(\lambda-t)}\Phi(\lambda-t) \\
& [C(\pi) + \frac{1}{\epsilon}U_\epsilon(\lambda) - \frac{1}{\epsilon}(U_\epsilon(\lambda) - \varsigma(\lambda))^+]d\lambda
\end{aligned} \tag{3.25}$$

Let us define the transformation T_ϵ of $C(0,T;C_1)$ into itself using the right hand side of (3.25). Then the latter can be written as a fixed point equation

$$U_\epsilon = T_\epsilon U_\epsilon \tag{3.26}$$

Using (3.11) and (3.15) one can show precisely as in Bensoussan-Lions [9, p.488] that some power of T_ϵ is a contraction. Hence the result of the lemma follows.

One then can also prove as in [9, pp.489 - 490], that if $\epsilon \leq \epsilon', \|U_\epsilon\|_1 \leq K$, then $0 \leq U_\epsilon \leq U_{\epsilon'}$. As in [9, pp.494 - 495] one then shows that as $\epsilon \downarrow 0, U_\epsilon \downarrow U$ which is the maximum element of (3.22). The convergence takes place in $C(0,T;C_1)$. This establishes Proposition 3.1.

We can then proceed with the

Proof of Theorem 3.1: (Continuation)

Let us consider the map H mapping $C(0,T;C_1) \times C(0,T;C_1)$ into itself defined by

$$H(\varsigma_1, \varsigma_2) = (U_1, U_2) \tag{3.27}$$

where the right hand side represents the maximum element of (3.20). Let now

$$\begin{aligned} U_1^o(\pi,t) &= \Phi_1(T-t)\Psi(\pi) + \int_t^T \Phi_1(\lambda - t)C_1(\pi)d\lambda \\ U_2^o(\pi,t) &= \Phi_2(T-t)\Psi(\pi) + \int_t^T \Phi_2(\lambda - t)C_2(\pi)d\lambda \end{aligned} \tag{3.28}$$

Consider $\varsigma_i(t), \xi_i(t), i = 1,2$ such that

$$0 \leq \varsigma_i(t) \leq \xi_i(t) \leq U_i^o(t), i = 1,2, \tag{3.29}$$

and

$$\xi_i(t) - \varsigma_i(t) \leq \gamma \xi_i(t), \ \gamma \in [0,1]. \tag{3.30}$$

Then we have

$$0 \leq H(\xi_1, \xi_2) - H(\varsigma_1, \varsigma_2) \leq \gamma(1-\gamma')H(\xi_1, \xi_2) \tag{3.31}$$

where

$$\gamma' \leq \frac{k_0}{k_0 + \overline{\Psi} + \max(\|C_1\|, \|C_2\|)T} \tag{3.32}$$

Indeed, setting

$$\kappa = 1 - \gamma(1-\gamma') \tag{3.33}$$

we have to prove that

$$\kappa H(\xi_1, \xi_2) \leq H(\varsigma_1, \varsigma_2). \tag{3.34}$$

Let us set

$$\begin{aligned} (U_1, U_2) &= H(\varsigma_1, \varsigma_2) \\ (\tilde{U}_1, \tilde{U}_2) &= H(\xi_1, \xi_2). \end{aligned} \tag{3.35}$$

We need then to show that

$$\kappa \tilde{U}_1 \leq U_1, \qquad \kappa \tilde{U}_2 \leq U_2. \tag{3.36}$$

If we can establish that

$$\begin{aligned} \kappa K_1(\pi) + \kappa \xi_2(\pi,t) &\leq K_1(\pi) + \varsigma_2(\pi,t) \\ \kappa K_2(\pi) + \kappa \xi_1(\pi,t) &\leq K_2(\pi) + \varsigma_1(\pi,t), \end{aligned} \tag{3.37}$$

then (3.36) is implied by the monotonicity properties of Variational Inequalities. But

$$\xi_2(\pi,t)(1-\gamma) \leq \varsigma_2(\pi,t), \tag{3.38}$$

hence it is enough to establish that

$$\begin{aligned} \kappa K_1(\pi) + \kappa \xi_2(\pi,t) &\leq K_1(\pi) + (1-\gamma)\,\xi_2(\pi,t) \\ \kappa K_2(\pi) + \kappa \xi_1(\pi,t) &\leq K_2(\pi) + (1-\gamma)\xi_1(\pi,t) \end{aligned} \tag{3.39}$$

The first of (3.39) will be satisfied if

$$[\kappa - (1-\gamma)]\xi_2(\pi,t) \leq (1-\kappa)K_1(\pi) \tag{3.40}$$

or if

$$\gamma'\xi_2(\pi,t) \leq (1-\gamma')K_1(\pi). \tag{3.41}$$

But observe that

$$\xi_2(\pi,t) \leq U_2^0(\pi,t) \leq (\overline{\Psi} + \|C_2\|T)\,(\pi,\mathbb{1})$$

So it is enough to choose γ' so that

$$\gamma'(\overline{\Psi} + \|C_2\|T)(\pi,\mathbb{1}) \leq (1-\gamma')k_0(\pi,\mathbb{1}) \tag{3.42}$$

where k_0 is the uniform lower bound (2.21), since $K_1(\pi) \geq k_0(\pi,\mathbb{1})$. This last inequality requires

$$\gamma' \leq \frac{k_0}{k_0 + \overline{\Psi} + \|C_2\|T} \tag{3.43}$$

In an identical fashion, the second of (3.39) will be satisfied if

$$\gamma' \leq \frac{k_0}{k_0 + \overline{\Psi} + \|C_1\|T}. \tag{3.44}$$

So both of (3.39) will be satisfied if we choose γ' according to (3.32). The proof of the theorem then proceeds via the standard iteration

$$(U_1^{n+1}, U_2^{n+1}) = H\,(U_1^n, U_2^n) \tag{3.45}$$

as in [9, pp.512 - 514].

Remark: *The extension of this result to the general case $N \neq 2$ is straightforward.* The system (3.10) has N functionals $U_1, \ldots, U_N$. Everything in (3.10) is the same except for the last two inequalities which are replaced by

$$U_i(\pi,t) \leq \min_{\substack{j\neq i \\ j=1,\ldots,N}} (K_{ij}(\pi) + U_j(\pi,t)), \qquad i = 1,\ldots,N \tag{3.46}$$

One again introduces the system (3.20) where the last two inequalities are replaced by

$$U_i(\pi,t) \le \min_{\substack{j\neq i \\ j=1,\ldots,N}} (K_{i\,j}(\pi) + \varsigma_j(\pi,t)), \qquad i = 1,\ldots,N \tag{3.47}$$

where $\varsigma_i \in C(0,T;C_1)$, and satisfy the remainder of (3.21). One then establishes the analog of Proposition 3.1 by penalization. The analog of Theorem 3.1 is established by introducing a map H mapping $C(0,T;C_1)^N$ into itself defined by

$$H(\varsigma_1,\varsigma_2,\ldots,\varsigma_N) = (U_1,U_2\ldots,U_N)$$

where the right hand side is the maximum element of the analog of (3.20).

3.3.3 Existence of an admissible sensor schedule

Our objective in this section is to show that the maximum element U_1, U_2 of the QVI (3.10) provides the value function for the optimization problem (2.61), (2.66) when the assumption (3.11) holds. Furthermore we want to show how an admissible optimal sensor schedule is determined once the pair U_1, U_2 is known.

We shall prove that

$$U_i(\pi,0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J(u(\cdot)), \qquad i = 1,2 \tag{3.48}$$

where $\pi \in H^+$ satisfies $(\pi, \mathbf{1}) = 1$. An optimal schedule will be constructed as follows. Suppose, to fix ideas that $i = 1$. Then define

$$\tau_1^* = \inf_{t\le T} \{U_1(p_1(t),t) = K_1(p_1(t)) + U_2(p_1(t),t)\} \tag{3.49}$$

where again $p_i(t)$ is the solution of (3.2). We write

$$p^*(t) = p_1(t), \qquad t \in [0,\tau_1^*]. \tag{3.50}$$

Next define

$$\tau_2^* = \inf_{\tau_1^*\le t\le T} \{U_2(p_2(t),t), = K_2(p_2(t)) + U_1(p_2(t),t)\} \tag{3.51}$$

In (3.51), it must be kept in mind that $p_2(t)$ represents the solution of (3.2) with j=2, starting at τ_1^* with value $p_1(\tau_1^*)$. We then define

$$p^*(t) = p_2(t), \qquad t \in [\tau_1^*,\tau_2^*] \tag{3.52}$$

Note that, unless $\tau_1^* = T$,

$$\tau_2^* > \tau_1^*, \tag{3.53}$$

otherwise

$$\begin{aligned}U_1(p_1(\tau_1^*),\tau_1^*) &= K_1(p_1(\tau_1^*)) + U_2(p_1(\tau_1^*),\tau_1^*)\\ U_2(p_1(\tau_1^*),\tau_1^*) &= K_2(p_1(\tau_1^*)) + U_1(p_1(\tau_1^*),\tau_1^*)\end{aligned} \tag{3.54}$$

which is impossible since

$$K_1(p_1(\tau_1^*)) > 0, K_2(p_1(\tau_1^*)) > 0 \qquad a.s. \tag{3.55}$$

Similarly one proceeds to construct a sequence of $\tau_1^* < \tau_2^* < \tau_3^* < \ldots$ and the process $p^*(\cdot)$. We can then prove the following.

Theorem 3.2. *With the same assumptions as in Theorem 3.1 and in addition assuming that (3.11) holds; the sequence of stopping times $\tau_1^*, \tau_2^*, \ldots$ defines an optimal admissible sensor schedule.*

Proof: Considering (3.10) as a VI with obstacle ς_2, ς_1, we can write from the definition of τ_1^*

$$\begin{aligned}U_1(\pi,0) &= E\{U_1(p_1(\tau_1^*),\tau_1^*)\\ &+ \int_0^{\tau_1^*} C_1(p_1(\lambda))d\lambda\}.\end{aligned} \tag{3.56}$$

This can be established by utilizing the penalization (3.23), along similar lines as in [9, pp. 578 - 587]. Then

$$\begin{aligned}E\{U_1(p_1(\tau_1^*),\tau_1^*)\} &= E\{U_1(p^*(\tau_1^*),\tau_1^*\}\\ &= E\{\Psi(p^*(T))\chi_{\tau_1^*=T}\}\\ &+ E\{U_1(p^*(\tau_1^*),\tau_1^*)\chi_{\tau_1^*<T}\}.\end{aligned}$$

Substituting back in (3.56) and using the definition of τ_1^* in (3.49) we obtain

$$\begin{aligned}U_1(\pi,0) &= E\Big\{\Psi(p^*(T))\chi_{\tau_1^*=T} + \int_0^{\tau_1^*} C_1(p^*(\lambda))d\lambda\\ &\quad +K_1(p^*(\tau_1^*))\chi_{\tau_1^*<T} + U_2(p^*(\tau_1^*),\tau_1^*)\chi_{\tau_1^*<T}\Big\}\end{aligned} \tag{3.57}$$

Furthermore, again by employing penalization one can show that

$$\begin{aligned}E\{U_2(p^*(\tau_1),\tau_1^*)\} = E\{U_2(p_2(\tau_1^*),\tau_1^*)\} &= E\{U_2(p_2(\tau_2^*),\tau_2^*)\\ &+ \int_{\tau_1^*}^{\tau_2^*} C_2(p_2(\lambda))d\lambda\}.\end{aligned} \tag{3.58}$$

This implies

$$\begin{aligned}E\{U_2(p_2(\tau_1^*),\tau_1^*)\chi_{\tau_1^*<T}\} &= E\{U_2(p_2(\tau_2^*),\tau_2^*)\chi_{\tau_1^*<T}\\ &+ \chi_{\tau_1^*<T}\int_{\tau_1^*}^{\tau_2^*} C_2(p_2(\lambda))d\lambda\}.\end{aligned} \tag{3.59}$$

Next

$$
\begin{aligned}
E\{U_2(p_2(\tau_2^*),\tau_2^*)\chi_{\tau_1^*<T}\} &= E\{\Psi(p^*(T))\chi_{\tau_1^*<T,\tau_2^*=T}\} \\
&+ E\{U_2(p^*(\tau_2^*),\tau_2^*)\chi_{\tau_2^*<T}\}.
\end{aligned}
$$

Substituting back in (3.57) and using the definition of τ_2^* in (3.51) we obtain

$$
\begin{aligned}
U_1(\pi,0) &= E\{\Psi(p^*(T))\chi_{\tau_2^*=T} + K_1(p^*(\tau_1^*))\chi_{\tau_1^*<T} \\
&+K_2(p^*(\tau_2^*))\chi_{\tau_2^*<T} + \int_0^{\tau_1^*} C_1(p^*(\lambda))d\lambda \\
&+\int_{\tau_1^*}^{\tau_2^*} C_2(p^*(\lambda))d\lambda + U_1(p^*(\tau_2^*),\tau_2^*)\chi_{\tau_2^*<T}\}
\end{aligned}
\tag{3.60}
$$

Proceeding in a similar fashion, and collecting results we can write:

$$
\begin{aligned}
U_1(\pi,0) &= E\{\Psi(p^*(T))\chi_{\tau_n^*=T} \\
&+ \sum_{i=1}^{n} K_i(p^*(\tau_i^*))\chi_{\tau_i^*<T} \\
&+ \sum_{i=0}^{n-1} \chi_{\tau_{i+1}^*<T}\int_{\tau_i^*}^{\tau_{i+1}^*} C_{i+1}(p^*(\lambda))d\lambda\}, \\
&+ U_{n+1}(p^*(\tau_n^*),\tau_n^*)\chi_{\tau_n^*<T}
\end{aligned}
\tag{3.61}
$$

where we used the notation

$$
\begin{aligned}
K_i &= \begin{cases} K_1, & \text{if i is odd} \\ K_2, & \text{if i is even} \end{cases} \\
C_i &= \begin{cases} C_1, & \text{if i is odd} \\ C_2, & \text{if i is even} \end{cases} \\
U_i &= \begin{cases} U_1, & \text{if i is odd} \\ U_2, & \text{if i is even.} \end{cases}
\end{aligned}
\tag{3.62}
$$

However, observe that necessarily $\tau_n^* = T$, for n large enough (random). Otherwise one has $\tau_n^* < T, \forall n$, on a set $\Omega_0 \subset \Omega$ of positive probability. But $\tau_n^* \uparrow \tau^* \leq T$ and

$$
(p^*(\tau_i^*),\mathbf{1}) \longrightarrow (p^*(\tau^*),\mathbf{1}) \tag{3.63}
$$

where (since $(\pi,\mathbf{1}) = 1$)

$$
(p^*(\tau^*),\mathbf{1}) = 1 + \int_0^{\tau^*} p^*\delta^T dy \tag{3.64}
$$

(see (2.66)) and

$$
(p^*(\tau^*),\mathbf{1}) = E\{\varsigma(\tau^*)|\mathcal{F}_{\tau^*}^{y(\cdot,u^*)}\} > 0 \quad a.s. \tag{3.65}
$$

where $\varsigma(\cdot)$ is the process introduced by (2.8). Therefore on Ω_0, as $n \to \infty$

$$
\sum_{i=1}^{n} K_i(p^*(\tau_i^*))\chi_{\tau_i^*<T} \longrightarrow +\infty \tag{3.66}
$$

and since Ω_0 has positive probability, as $n \to \infty$

$$E\{\sum_{i=1}^{n} K_i(p^*(\tau_i^*))\chi_{\tau_i^*<T}\} \longrightarrow \infty, \tag{3.67}$$

which contradicts (3.19).

We can thus assert that

$$\chi_{\tau_n^*=T} \longrightarrow 1 \qquad a.s. \tag{3.68}$$

In particular, it follows that the sequence $\tau_1^*, \tau_2^* \ldots,$ defines an admissible schedule denoted by u^*. The corresponding state solution of (2.66), coincides with p^* and (3.61) implies

$$U_1(\pi,0) \geq J(u^*(\cdot)). \tag{3.69}$$

But by standard arguments, one checks that

$$U_1(\pi,0) \leq J(u(\cdot)), \ \forall u(\cdot) \in U_{ad} \tag{3.70}$$

and therefore $u^*(\cdot)$ is indeed optimal.

3.3.4 The main result

We want now to get rid of (3.11) and consider the original functional Ψ in (3.8). Let us consider the approximation (3.12) Ψ_n of Ψ. To Ψ_n corresponds a system of QVI.

$$\begin{aligned}
U_1^n, U_2^n &\in C(0,T;\tilde{C}_1) \\
U_1^n, U_2^n &\geq 0 \\
U_1^n(\pi,T) &= U_2^n(\pi,T) = \Psi_n(\pi) \\
U_1^n(\pi,t) &\leq \Phi_1(s-t)U_1^n(\pi,s) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda \\
U_2^n(\pi,t) &\leq \Phi_2(s-t)U_2^n(\pi,s) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \\
\forall s &\geq t \\
U_1^n(\pi,t) &\leq K_1(\pi) + U_2^n(\pi,t) \\
U_2^n(\pi,t) &\leq K_2(\pi) + U_1^n(\pi,t).
\end{aligned} \tag{3.71}$$

From Theorem 3.2, we can assert that

$$U_i^n(\pi,0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J^n(u(\cdot)), \qquad i = 1,2 \tag{3.72}$$

where

$$\begin{aligned}
J^n(u(\cdot)) &= E\{\Psi^n(p(u(\cdot),T)) + \int_0^T (p(u(\cdot),t), C(u(t)))dt \\
&+ \sum_{i=1}^{\infty} \chi_{\tau_i<T}(p(u(\cdot),\tau_i), K(u_{i-1},u_i))\}.
\end{aligned} \tag{3.73}$$

Therefore we deduce that

$$J^n(u(\cdot)) - J(u(\cdot)) = E\{\Psi_n(p(u(\cdot),T)) - \Psi(p(u(\cdot),T))\} \tag{3.74}$$

and from (3.12) we deduce

$$\begin{aligned} |J^n(u(\cdot)) - J(u(\cdot))| \leq\ & E\left\{\int \frac{p(u(\cdot),T)\|x\|^4}{n+\|x\|^2}dx\right\} \\ & +E\left\{\left(\int p(u(\cdot),T)x\left(1-\frac{1}{(1+\frac{\|x\|^2}{n})^{1/2}}\right)dx\right)^T\right. \\ & \left.\cdot\left(\int p(u(\cdot),T)x(1+\frac{1}{(1+\frac{\|x\|^2}{n})^{1/2}})dx\right) \times \frac{1}{\int p(u(\cdot),T)dx}\right\} \end{aligned} \tag{3.75}$$

But using the equation (2.50) yields (see (2.1a))

$$\begin{aligned} E\left\{\int \frac{p(u(\cdot),t)\|x\|^4}{n+\|x\|^2}dx\right\} = & E\left\{\int_0^t \int p(u(\cdot),s)(x)\left\{\frac{\partial a_{ij}}{\partial x_i}\frac{2\|x\|^2(2n+\|x\|^2)x_j}{(n+|x|^2)^2}\right.\right. \\ & +a_{ij}\left(\delta_{ij}\frac{2\|x\|^2(2n+\|x\|^2)}{(n+\|x\|^2)^2}+\frac{8x_ix_jn^2}{(n+\|x\|^2)^3}\right) \\ & \left.\left. -a_i\frac{2\|x\|^2(2n+\|x\|^2)x_i}{(n+\|x\|^2)^2}\right\}ds + \int\frac{\pi(x)\|x\|^4}{n+\|x\|^2}dx\right\} \end{aligned}$$

where we employed the summation convention over repeated indices. Hence after majorizing conveniently

$$\begin{aligned} E\left\{\int \frac{p(u(\cdot),t)(x)\|x\|^4}{n+\|x\|^2}dx\right\} \leq\ & \int\frac{\pi(x)\|x\|^4}{n+\|x\|^2}dx \\ & + \Gamma\int_0^t E\left\{\int \frac{p(u(\cdot),s)(x)\|x\|^4}{n+\|x\|^2}dx\right\}ds + \frac{\Gamma t}{n}. \end{aligned} \tag{3.76}$$

We shall use capital Greek letters, $\Gamma, \Delta, \ldots$, to indicate constants in the following estimates. Finally we deduce

$$\begin{aligned} E\left\{\int \frac{p(u(\cdot),t)(x)\|x\|^4}{n+\|x\|^2}dx\right\} & \leq \Gamma_t\left[\int\frac{\pi(x)\|x\|^4}{n+\|x\|^2}dx+\frac{1}{n}\right] \\ & \leq \Gamma_t\left[\frac{1}{n}\int\pi(x)\|x\|^4dx+\frac{1}{n}\right]. \end{aligned} \tag{3.77}$$

Next consider

$$\frac{p(u(\cdot),t)}{(p(u(\cdot),t),\mathbb{1})} = \sigma(u(\cdot),t)$$

which is the normalized conditional probability, measure and satisfies Kusner's equation

$$d(\sigma(t)(\varphi)) = \sigma(t)(L\varphi)dt + (\sigma(t)(\bar{h}\varphi) - \sigma(t)(\varphi)\sigma(t)(\bar{h}))\cdot(dz - \sigma(t)(\bar{h})dt) \tag{3.78}$$

If we apply (3.78) with $\varphi = \|x\|^2 = \chi^2$, we obtain

$$\begin{aligned} dE\{\sigma(t)(\chi^2)\} &= E\{\sigma(t)(L\chi^2) - \sigma(t)(\tilde{h})[\sigma(t)(\tilde{h}\chi^2) - \sigma(t)(\chi^2)\sigma(\tilde{h})]\}\, dt \\ &\le \Delta_0(1 + E\{\sigma(t)(\chi^2)\}). \end{aligned} \tag{3.79}$$

Finally

$$E\{\sigma(t)(\chi^2)\} \le \Delta_t \int \pi(x)\|x\|^2 dx \tag{3.80}$$

But the 2nd term in (3.75) is

$$E\left\{\sigma(T)\left(\chi(1 + \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)^T (p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\right\}$$

$$\begin{aligned} &\le \left[E\left\{\|\sigma(T)\left(\chi(1 + \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right]^{1/2} \left[E\left\{\|p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right]^{1/2} \\ &\le \Delta^1(E\{\sigma(T)(\chi^2)\})^{1/2}\left(E\left\{\|p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right)^{1/2} \\ &\le \Delta^2\left[E\left\{\|p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right]^{1/2} \\ &= \Delta^3\left[E\left\{\sum_i\left(p(T)\left(\chi_i(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\right)^2\right\}\right]^{1/2} \\ &\le \Delta^3\left[E\left\{p(T)(\chi^2)p(T)\left(\frac{\chi^2}{n+\chi^2}\right)\right\}\right]^{1/2}. \end{aligned} \tag{3.81}$$

One easily checks that

$$\begin{aligned} E\left\{\left(p(T)(\chi^2)\right)^2\right\} &\le \Delta^4 + \left(\int \pi(x)\|x\|^2 dx\right)^2 \le \Delta^5 \\ dE\left\{|p(t)\left(\frac{\chi^2}{n+\chi^2}\right)|^2\right\} &\le 2E\left\{p(t)\left(L\frac{\chi^2}{n+\chi^2}\right)p(t)(\frac{\chi^2}{n+\chi^2})\right\} dt \\ &+ \Delta^6 E\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|^2\right\} dt \end{aligned}$$

But

$$L\frac{\chi^2}{n+\chi^2} \le \frac{\Delta^6}{\sqrt{n}} \tag{3.82}$$

hence

$$dE\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|^2\right\} \le \left[\Delta^6 E\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|^2\right\} + \frac{\Delta^7}{n}\right] dt \tag{3.83}$$

which implies

$$E\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|\right\} \leq \Theta_t\left[\frac{1}{n}+\left(\int\frac{\pi(x)\|x\|^2dx}{n+\|x\|^2}\right)^2\right] \leq \frac{\Theta_t}{n}(1+\int\pi(x)\|x\|^4dx) \tag{3.84}$$

Therefore, continuing from (3.81), the 2nd term in (3.75) is majorized by $\frac{\Gamma_0}{n^{1/4}}$. Collecting results (from (3.75), (3.77), (3.81), (3.84)) we can assert that

$$|J^n(u(\cdot)) - J(u(\cdot))| \leq \frac{\Delta}{n^{1/4}} \tag{3.85}$$

provided the initial distribution of $p(0)$, i.e. π satifies

$$\int\pi(x)\|x\|^4dx < \infty \tag{3.86}$$

The estimate in (3.85) is uniform with respect to n. Therefore

$$|U_i^n(\pi,0) - \inf_{\substack{u(0)=i\\p(0)=\pi}} J(u(\cdot))| \leq \frac{\Delta}{n^{1/4}} \tag{3.87}$$

In fact we can replace 0 by any $t\in[0,T]$ and consider the function

$$U_i(\pi,t) = \inf_{\substack{u(t)=i\\p(t)=\pi}} J_t(u(\cdot)) \tag{3.88}$$

where $J_t(u(\cdot))$ corresponds to a problem analogous to (2.50), (2.61) starting in t instead of 0. Therefore we have

$$|U_i^n(\pi,t) - U_i(\pi,t)| \leq \frac{\Delta}{n^{1/4}} \tag{3.89}$$

We have however to be careful to the fact that the constant in (3.89) depends on a bound on $\int\pi(x)\|x\|^4dx$. More precisely we have proved that

$$|U_i^n(\pi,t) - U_i(\pi,t)| \leq \frac{\Delta'}{n^{1/4}}(1+\int\pi(x)\|x\|^4dx) \tag{3.90}$$

where Δ' this time does not depend on π (assuming that π is a probability). It follows that

$$U_i^n(\pi,t) \longrightarrow U_i(\pi,t) \text{ in } C(0,T;C_1). \tag{3.91}$$

Taking the limit in (3.71), we obtain that U_1,U_2 is a solution of (3.10) and moreover

$$U_i(\pi,0) = \inf_{\substack{u(0)=i\\p(0)=\pi}} J(u(\cdot)) \tag{3.92}$$

However by a probabilistic argument already used in section 3.3, any solution of (3.10) is smalller than the right hand side of (3.92). This completes the proof of Theorem 3.1, and also provides the same statement as in Theorem 3.2, without the assumption (3.11) and for our original Ψ given by (3.8).

References

[1] M. Hazewinkel and J.C. Willems, (Eds), *Stochastic Systems: The Mathematics of Filtering and Identification and Applications*, Proceedings of NATO Advanced Study Institute, Les Arcs, France, Reidel 1981.

[2] W.H. Fleming and L.G. Gorostiza, (Eds), *Advances in Filtering and Optimal Stochastic Control*, Proc. IFIP-WG 7/1 Working Conference, Cocoyoc, Mexico, 1982, Lect. Notes Control and Inf. Sciences, No. 42, Springer-Verlag, Berlin, 1982.

[3] M. Metivier and E. Pardoux, (Eds), *Stochastic Differential Systems: Filtering and Control*, Proc. IFIP-WG 7/1 Working Conference, Marseille-Luminy, France, 1984, Lect. Notes Control and Inf. Sciences, No. 69, Springer-Verlag, Berlin, 1985.

[4] W.H. Fleming and E. Pardoux, "Optimal Control for Partially Observed Diffusions", *SIAM J. Control and Optim.*, vol. 20, No. 2, March 1982, pp. 261-285.

[5] J.M. Bismut, "Partially Observed Diffusions and Their Control", *Ibid.*, vol. 20, No. 2, March 1982, pp. 302-309.

[6] John S. Baras, "Optimal Sensor Scheduling in Multiple Sensor Platforms", in preparation.

[7] J.D. Kattar, "A Solution of the Multi-Weapon, multi-target Assignment Problem", WP-26597, The MITRE Co , Bedford, MA, Feb., 1986.

[8] G. Kallianpur, *Stochastic Filtering Theory*, Springer-Verlag, New York, 1980.

[9] A. Bensoussan and J.L. Lions, *Contrôle Impulsionnel et Inéquations Quasi-variationelles*, Dunod, Paris, 1982.

[10] A. Bensoussan, "Maximum Principle and Dynamic Programming Approaches of the Optimal Control of Partially Observed Diffusions", *Stochastics*, 1983, vol 9, pp 169-222.

[11] E. Pardoux, "Stochastic Partial Differential Equations and Filtering of Diffusion Processes", *Stochastics*, vol 9, no. 3, (1983).

[12] J.L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, vols. 1, 2, Dunod, Paris 1968, English translation, Springer-Verlag, 1972.

[13] John S. Baras and Alain Bensoussan, "Optimal Sensor Scheduling in Nonlinear Filtering of Diffusion Processes II: Computational Methods", in preparation.

[14] R.S. Liptser and A.N. Shiryayev, *Statistics of Random Processes I: General Theory*, Springer-Verlag, Berlin, 1977.

4 The Partially Observed Stochastic Minimum Principle

4.1 Introduction

Various proofs have been given of the minimum principle satisfied by an optimal control in a partially observed stochastic control problem. See, for example, the papers by Bensoussan [1], Elliott [5], Haussmann [7], and the recent paper [9] by Haussmann in which the adjoint process is identified. The simple case of a partially observed Markov chain is discussed in the University of Maryland lecture notes [6] of the second author.

We show in this article how a minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. The results of Bismut [2], [3] and Kunita [10], on stochastic flows enable us to compute in an easy and explicit way the change in the cost due to a 'strong variation' of an optimal control. The only technical difficulty is the justification of the differentiation. As we wished to exhibit the simplification obtained by using the ideas of stochastic flows the result is not proved under the weakest possible hypotheses. Finally, in Section 6, we show how Bensoussan's minimum principle follows from our result if the drift coefficient is differentiable in the control variable.

4.2 Dynamics

Suppose the state of the system is described by a stochastic differential equation

$$d\xi_t = f(t, \xi_t, u)dt + g(t, \xi_t)dw_t,$$

$$\xi_t \in R^d, \quad \xi_0 = x_0, \quad 0 \leq t \leq T. \tag{2.1}$$

The control parameter u will take values in a compact subset U of some Euclidean space R^k. We shall make the following assumptions:

A_1: x_0 is given; if x_0 is a random variable and P_0 its distribution the situation when $\int |x|^q P_0(dx) < \infty$ for some $q > n + 1$ can be treated, as in [9], by including an extra integration with respect to P_0.

A_2: $f : [0,T] \times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t,x), continuously differentiable in x and for some constant K

$$(1 + |x|)^{-1} \; |f(t,x,u)| + |f_x(t,x,u)| \leq K_1.$$

A_3: $g : [0,T] \times R^d \to R^d \otimes R^n$ is a matrix valued function, Borel measurable, continuously differentiable in x, and for some constant K_2

$$|g(t,x)| + |g_x(t,x)| \leq K_2.$$

The observation process is given by

$$dy_t = h(\xi_t)dt + dv_t$$

$$y_t \in R^m, \quad y_0 = 0, \quad 0 \leq t \leq T. \tag{2.2}$$

In the above equations $w = (w^1, \ldots, w^n)$ and $v = (v^1, \ldots, v^d)$ are independent Brownian motions. We also assume

A_4: $h : R^d \to R^m$ is Borel measurable, continuously differentiable in x, and for some constant K_3

$$|h(t,x)| + |h_x(t,x)| \leq K_3.$$

REMARKS 2.1. These hypotheses can be weakened. For example, in A_4, h can be allowed linear growth in x. Because g is bounded a delicate argument then implies the exponential Z of (2.3) is in some L^p space, $1 < p < \infty$. (See, for example, Theorem 2.2 of [8]). However, when h is bounded Z is in all the L^p spaces, (see Lemma 2.3). Also, if we require f to have linear growth in u then the set of control values U can be unbounded as in [9]. Our objective, however, is not the greatest generality but to demonstrate the simplicity of the techniques of stochastic flows.

Let $\hat{P}$ denote Wiener measure on the $C([0,T],R^n)$ and μ denote Wiener measure on $C([0,T],R^m)$. Consider the space $\Omega = C([0,T],R^n) \times C([0,T],R^m)$ with coordinate functions (x_t, y_t) and define Wiener measure P on Ω by

$$P(dx,dy) = \hat{P}(dx)\mu(dy).$$

DEFINITION 2.2. *Write $Y = \{Y_t\}$ for the right continuous complete filtration on $C([0,T],R^m)$ generated by $Y_t^0 = \sigma\{y_s : s \leq t\}$. The set of admissible control functions $\underline{U}$ will be the Y-predictable functions on $[0,T] \times C([0,T],R^m)$ with values in U.*

For $u \in \underline{U}$ and $x \in R^d$ write $\xi^u_{s,t}(x)$ for the strong solution of (2.1) corresponding to control u, and with $\xi^u_{s,s}(x) = x$. Write

$$Z^u_{s,t}(x) = \exp\left(\int_s^t h(\xi^u_{s,r}(x))' dy_r - \frac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2 dt\right) \tag{2.3}$$

and define a new probability measure P^u on Ω by $\frac{dP^u}{dP} = Z^u_{0,T}(x_0)$. Then under P^u $(\xi^u_{0,t}(x_0), y_t)$ is a solution of (2.1) and (2.2), that is $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion v such that y_t satisfies (2.2). A version of Z defined for every trajectory y of the observation process is obtained by integrating by parts the stochastic integral in (2.3).

LEMMA 2.3. *Under hypothesis A_4, for $t \leq T$,*

$$E[(Z^u_{0,t}(x_0))^p] < \infty \quad \textit{for all } u \in \underline{U} \textit{ and all } p, \quad 1 \leq p < \infty.$$

PROOF.

$$Z^u_{0,t}(x_0) = 1 + \int_0^t Z^u_{0,r}(x_0) h(\xi^u_{0,r}(x_0))' dy_r.$$

Therefore, for any p there is a constant C_p such that

$$E\Big[\big(Z^u_{0,t}(x_0)\big)^p\Big] \le C_p\Big[1 + E\Big(\int_0^t \big(Z^u_{0,r}(x_0)\big)^2 h\big(\xi^u_{0,r}(x_0)\big)^2 dr\Big)^{p/2}\Big].$$

The result follows by Gronwall's inequality.

COST 2.4. We shall suppose the cost is purely terminal and given by some bounded, differentiable function

$$c\big(\xi^u_{0,T}(x_0)\big)$$

which has bounded derivatives. Then the expected cost if control $u \in \underline{U}$ is used is

$$J(u) = E_u\Big[c\big(\xi^u_{0,T}(x_0)\big)\Big].$$

In terms of P, under which y_t is always a Brownian motion, this is

$$J(e) = E\Big[Z^u_{0,T}(x_0)\, c\big(\xi^u_{0,T}(x_0)\big)\Big]. \tag{2.4}$$

4.3 Stochastic Flows

For $u \in \underline{U}$ write

$$\xi^u_{s,t}(x) = x + \int_s^t f(r,\ \xi^u_{s,r}(x),\ u_r)dr + \int_s^t g(r,\ \xi^u_{s,r}(x))dw_r \tag{3.1}$$

for the solution of (2.1) over the time interval $[s,t]$ with initial condition $\xi^u_{s,s}(x) = x$. In the sequel we wish to discuss the behaviour of (3.1) for each trajectory y of the observation process. We have already noted there is a version of Z defined for every y. The results of Bismut [2] and Kunita [10] extend easily and show the map

$$\xi^u_{s,t} : R^d \to R^d$$

is, almost surely, for each $y \in C([0,T], R^m)$ a diffeomorphism. Bismut [2] initially gives proofs when the coefficients f and g are bounded, but points out that a stopping time argument extends the results to when, for example, the coefficients have linear growth.

Write $\|\xi^u(x_0)\|_t = \sup_{0 \le s \le t} |\xi^u_{0,s}(x_0)|$. Then, as in Lemma 2.1 of [8], for any p, $1 \le p < \infty$ using Gronwall's and Jensen's inequalities

$$\|\xi^u(x_0)\|^p_T \le C\Big(1 + |x_0|^p + \Big|\int_0^t g(r,\ \xi^u_{0,r}(x_0))dw_r\Big|^p\Big)$$

almost surely, for some constant C.

Therefore, using Burkholder's inequality and hypothesis A_3, $\|\xi^u(x_0)\|_T$ is in L^p for all p, $1 \le p < \infty$.

Suppose $u^* \in U$ is an optimal control so $J(u^*) \le J(u)$ for any other $u \in U$. Write $\xi^*_{s,t}(\cdot)$ for $\xi^{u^*}_{s,t}(\cdot)$. The Jacobian $\dfrac{\partial \xi^*_{s,t}}{\partial x}(x)$ is the matrix solution C_t of the equation for $s \le t$,

$$dC_t = f_x(t,\ \xi^*_{s,t}(x),\ u^*)C_t dt + \sum_{i=1}^n g_x^{(i)}(t,\ \xi^*_{s,t}(x))C_t dw^i_t \tag{3.2}$$
$$\text{with } C_s = I.$$

Here I is the $n \times n$ identity matrix and $g^{(i)}$ is the i^{th} column of g. From hypotheses A_2 and A_3, f_x and g_x are bounded. Writing $\|C\|_T = \sup_{0 \le s \le t} |C_s|$ an application of Gronwall's

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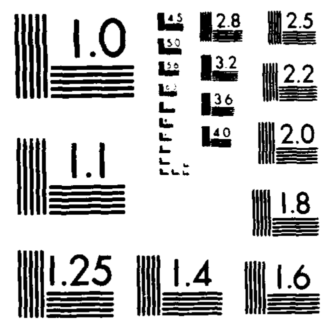

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Jensen's and Burkholder's inequalities again implies $\|C\|_T$ is in L^p for all p, $1 \leq p < \infty$. Consider the related matrix valued stochastic differential equation

$$
\begin{aligned}
D_t = I - \int_s^t D_r f_x(r,\ \xi^*_{s,r}(x),\ u^*_r)' dr \\
- \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r,\ \xi^*_{s,r}(x))' dw^i_r \\
+ \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r,\ \xi^*_{s,r}(x))')^2 dr.
\end{aligned}
\tag{3.3}
$$

Then it can be checked that $D_t C_t = I$ for $t \geq s$, so that D_t is the inverse of the Jacobian, that is $D_t = \left(\frac{\partial \xi^*_{s,t}(x)}{\partial x}\right)^{-1}$. Again, because f_x and g_x are bounded we have that $\|D\|_t$ is in every L^p, $1 \leq p < \infty$.

For a d-dimensional semimartingale z_t Bismut [2] shows one can consider the flow $\xi^*_{s,t}(z_t)$ and gives the semimartingale representation of this process. In fact if $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i dw^i_r$ is the d-dimensional semimartingale, Bismut's formula states that

$$
\begin{aligned}
\xi^*_{s,t}(z_t) = z_s + \int_s^t \Big(f(r,\ \xi^*_{s,r}(z_r),\ u^*_r) \\
+ \sum_{i=1}^n g_x^{(i)}(r,\ \xi^*_{s,r}(z_r),\ u^*_r) \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \xi^*_{s,r}(z_r)}{\partial x^2}(H_i, H_i) \Big) dr \\
+ \int_s^t \frac{\partial \xi^*_{s,r}(z_r)}{\partial x} dA_r + \sum_{i=1}^n \int_s^t \Big(g^{(i)}(r,\ \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \Big) dw^i_r.
\end{aligned}
\tag{3.4}
$$

DEFINITION 3.1. *We shall consider perturbations of the optimal control* u^* *of the following kind: For* $s \in [0,T)$, $h > 0$ *such that* $0 \leq s < s + h \leq T$, *for any other admissible control* $u \in \underline{U}$ *and* $A \in Y_s$ *define a strong variation of* u^* *by*

$$
u(t,w) = \begin{cases} u^*(t,w) & \text{if } (t,w) \notin [s, s+h] \times A \\ \tilde{u}(t,w) & \text{if } (t,w) \in [s, s+h] \times A. \end{cases}
$$

Applying (3.4) as in Theorem 5.1 of [4] we have the following result.

THEOREM 3.2. *For the perturbation* u *of the optimal control* u^* *consider the process*

$$
z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x}\right)^{-1} \Big(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(z_r),\ u^*_r) \Big) dr. \tag{3.5}
$$

Then the process $\xi^{*}_{s,t}(z_t)$ is indistinguishable from $\xi^{u}_{s,t}(x)$.

PROOF. Note the equation defining z_t involves only an integral in time; there is no martingale term, so to apply (3.4) we have $H_i = 0$ for all i. Therefore, from (3.4)

$$
\begin{aligned}
\xi^{*}_{s,t}(z_t) = x &+ \int_s^t f(r,\ \xi^{*}_{s,r}(z_r),\ u^{*}_r)dr \\
&+ \int_s^t \Big(\frac{\partial \xi^{*}_{s,r}(z_r)}{\partial x}\Big)\Big(\frac{\partial \xi^{*}_{s,r}(z_r)}{\partial x}\Big)^{-1} \big(f(r,\ \xi^{*}_{s,r}(z_r),\ u_r) - f(r,\ \xi^{*}_{s,r}(z_r),\ u^{*}_r\big)dr \\
&+ \int_s^t g(r,\ \xi^{*}_{s,r}(z_r))dw_r.
\end{aligned}
$$

However, the solution of (3.2) is unique so

$$\xi^{*}_{s,t}(z_t) = \xi^{u}_{s,t}(x).$$

REMARKS 3.3. Note that the perturbation $u(t)$ equals $u^{*}(t)$ if $t > s+h$ so $z_t = z_{s+h}$ if $t > s+h$ and

$$\xi^{*}_{s,r}(z_r) = \xi^{*}_{s,t}(z_{s+h}) = \xi^{*}_{s+h,t}(\xi^{u}_{s,s+h}(x)).$$

4.4 Augmented Flows

Consider the augmented flow which includes as an extra coordinate the stochastic exponential $Z^{*}_{s,t}$ with a 'variable' initial condition $z \in R$ for $Z^{*}_{s,s}(\cdot)$. That is, consider the $(d+1)$ dimensional system given by:

$$\xi^{*}_{s,t}(x) = x + \int_s^t f(r,\, \xi^{*}_{s,r}(x),\, u^{*}_r)dr + \int_s^t g(r, \xi^{*}_{s,r}(x))dw_r$$

$$Z^{*}_{s,t}(x,z) = z + \int_s^t Z^{*}_{s,r}(x,z)h(\xi^{*}_{s,r}(x))'dy_r.$$

Therefore,

$$\begin{aligned} Z^{*}_{s,t}(x,z) &= zZ^{*}_{s,r}(x) \\ &= z\exp\Big(\int_s^t h\big(\xi^{*}_{s,r}(x)\big)'dy_r - \frac{1}{2}\int_s^t h\big(\xi^{*}_{s,r}(x)\big)^2 dr\Big) \end{aligned}$$

and we see there is a version of the enlarged system defined for each trajectory y by integrating by parts the stochastic integral. The augmented map $(x,z) \to (\xi^{*}_{s,t}(x),\ Z^{*}_{s,t}(x,z))$ is then almost surely a diffeomorphism of R^{d+1}. Note that $\frac{\partial \xi^{*}_{s,t}(x)}{\partial z} = 0$, $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial g}{\partial z} = 0$. The Jacobian of this augmented map is, therefore, represented by the matrix

$$\widetilde{C}_t = \begin{pmatrix} \frac{\partial \xi^{*}_{s,t}(x)}{\partial x} & 0 \\ \frac{\partial Z^{*}_{s,t}(x,z)}{\partial x} & \frac{\partial Z^{*}_{s,t}(x,z)}{\partial z} \end{pmatrix},$$

and for $1 \le i \le d$ from equation (3.3)

$$\begin{aligned} \frac{\partial Z^{*}_{s,t}(x,z)}{\partial x_i} = \sum_{j=1}^{m}\int_s^t \Big(Z^{*}_{s,r}(x,z)\frac{\partial h^j(\xi^{*}_{s,r}(x))}{\partial \xi_k}\cdot\frac{\partial \xi^{*}_{k,s,r}(x)}{\partial x_i} \\ + h^j(\xi^{*}_{s,r}(x))\frac{dZ^{*}_{s,r}(x,z)}{\partial x_i}\Big)dy^j_r. \end{aligned} \tag{4.1}$$

(Here the double index k is summed from 1 to n).

We shall be interested in the solution of this differential system (4.1) only in the situation when $z = 1$ so we shall write $Z^{*}_{s,t}(x)$ for $Z^{*}_{s,t}(x,1)$. The following result is motivated by formally differentiating the exponential formula for $Z^{*}_{s,t}(x)$.

LEMMA 4.1.

$$\frac{\partial Z^*_{s,t}(x)}{\partial x} = Z^*_{s,t}(x)\Big(\int_s^t h_x(\xi^*_{s,r}(x))\cdot\frac{\partial \xi^*_{s,r}(x)}{\partial x}\cdot dv_r\Big)$$

where $v = (v^1,\dots,v^n)$ is the Brownian motion in the observation process.

PROOF. From (4.1) we see $\frac{\partial Z^*_{s,t}(x)}{\partial x}$ is the solution of the stochastic differential equation

$$\frac{\partial Z^*_{s,t}(x)}{\partial x} = \int_s^t \Big(\frac{\partial Z^*_{s,r}(x)}{\partial x}h'(\xi^*_{s,r}(x)) + Z^*_{s,r}(x)h_x(\xi^*_{s,r}(x))\frac{\partial \xi^*_{s,r}(x)}{\partial x}\Big)dy_r. \qquad (4.2)$$

Write

$$L_{s,t}(x) = Z^*_{s,t}(x)\Big(\int_s^t h_x\cdot\frac{\partial \xi^*_{s,r}}{\partial x}\cdot dv_r\Big)$$

where

$$dy_r = h(\xi^*_{s,t}(x))dt + dv_t.$$

Because

$$Z^*_{s,t}(x) = 1 + \int_s^t Z^*_{s,r}(x)h'(\xi^*_{s,r}(x))dy_r$$

the product rule gives

$$\begin{aligned}L_{s,t}(x) &= \int_s^t Z^*_{s,r}(x)h_x\cdot\frac{\partial \xi^*_{s,r}}{\partial x}dv_r\\ &\quad+\int_s^t\Big(\int_s^r h_x\cdot\frac{\partial \xi^*_{s,\sigma}}{\partial x}\cdot dv_\sigma\Big)Z^*_{s,r}(x)h'(\xi^*_{s,r}(x))dy_r\\ &\quad+\int_s^t Z^*_{s,r}(x)h'(\xi^*_{s,r}(x))\cdot h_x\cdot\frac{\partial \xi^*_{s,r}}{\partial x}dr\\ &=\int_s^t L_{s,r}(x)h'(\xi^*_{s,r}(x))dy_r + \int_x^t Z^*_{s,r}(x)h_x\cdot\frac{\partial \xi^*_{s,r}}{\partial x}\cdot dy_r.\end{aligned}$$

Therefore, $L_{s,t}(x)$ is also a solution of (4.2) so by uniqueness

$$L_{s,t}(x) = \frac{\partial Z^*_{s,t}(x)}{\partial x}.$$

REMARKS 4.2. As noted at the beginning of this section we can consider the augmented flow

$$(x,z)\to(\xi^*_{s,t}(x),\ Z^*_{s,t}(x,z))\quad\text{for } x\in R^d,\ z\in R,$$

and we are only interested in the situation when $z = 1$, so we write $Z^*_{s,t}(x)$.

LEMMA 4.3. *$Z^{*}_{s,t}(z_t) = Z^{u}_{s,t}(x)$ where z_t is the semimartingale defined in (3.6).*

PROOF. $Z^{u}_{s,t}(x)$ is the process uniquely defined by

$$Z^{u}_{s,t}(x) = 1 + \int_s^t Z^{u}_{s,r}(x)h'(\xi^{u}_{s,r}(x))dy_r. \tag{4.2}$$

Consider an augmented $(d+1)$ dimensional version of (3.6) defining a semimartingale $\overline{z}_t = (z_t, 1)$, so the additional component is always identically 1. Then applying (3.5) to the new component of the augmented process we have

$$\begin{aligned} Z^{*}_{s,r}(z_r) &= 1 + \int_s^t Z^{*}_{s,r}(z_r)h'(\xi^{*}_{s,r}(z_r))dy_r \\ &= 1 + \int_s^t Z^{*}_{s,r}(z_r)h'(\xi^{u}_{s,r}(x))dy_r \end{aligned}$$

by Theorem 3.2. However, (4.2) has a unique solution so $Z^{*}_{s,t}(z_t) = Z^{u}_{s,t}(x)$.

REMARKS 4.4. Note that for $t > s + h$

$$Z^{*}_{s,t}(z_t) = Z^{*}_{s,t}(z_{s+h}).$$

4.5 The Minimum Principle

Control u will be the perturbation of the optimal control u^* as in Definition 3.1. We shall write $x = \xi^*_{0,s}(x_0)$. Then the minimum cost is

$$\begin{aligned} J(u^*) &= E[Z^*_{0,T}(x_0)c(\xi^*_{0,T}(x_0))] \\ &= E[Z^*_{0,s}(x_0)Z^*_{s,T}(x)c(\xi^*_{s,T}(x))]. \end{aligned}$$

The cost corresponding to the perturbed control u is

$$\begin{aligned} J(u) &= E[Z^*_{0,s}(x_0)Z^u_{s,T}(x)c(\xi^u_{s,T}(x))] \\ &= E[Z^*_{0,s}(x_0)Z^*_{s,T}(z_{s+h})c(\xi^*_{s,T}(z_{s+h}))] \end{aligned}$$

by Theorem 3.2 and Lemma 4.3. Now $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are differentiable with continuous and uniformly integrable derivatives. Therefore

$$\begin{aligned} J(u) - J(u^*) &= E[Z^*_{0,s}(x_0)(Z^*_{s,T}(z_{s+h})c(\xi^*_{s,t}(z_{s+h})) - Z^*_{s,T}(x)c(\xi^*_{s,T}(x)))] \\ &= E\Big[\int_s^{s+h} \Gamma(s, z_r)(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(x),\ u^*_r))dr\Big] \end{aligned}$$

where

$$\begin{aligned} \Gamma(s, z_r) = Z^*_{0,s}(x_0)Z^*_{s,T}(z_r)\Big\{&c_\xi(\xi^*_{s,T}(z_r))\frac{\partial \xi^*_{s,T}(z_r)}{\partial x} + \\ &c(\xi^*_{s,T}(z_r))\Big(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r))\frac{\partial \xi^*_{s,\sigma}}{\partial x}(z_r)dv_\sigma\Big)\Big\}\Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1}. \end{aligned}$$

Note that this expression gives an explicit formula for the change in the cost resulting from a variation in the optimal control. The only remaining problem is to justify differentiating the right hand side.

From Lemma 2.3, Z is in every L^p space, $1 \le p < \infty$ and from the remarks at the beginning of Section 3, $C_T = \frac{\partial \xi^*_{s,T}}{\partial x}$ and $D_T = \left(\frac{\partial \xi^*_{s,T}}{\partial x}\right)^{-1}$ are in every L^p space, $1 \le p < \infty$. Consequently, Γ is in every L^p space, $1 \le p < \infty$.

Therefore

$$
\begin{aligned}
J(u) - J(u^*) = & \int_s^{s+h} E\Big[\big(\Gamma(s,z_r) - \Gamma(s,x)\big)\big(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(z_r),\ u^*_r)\big)\Big]dr \\
& + \int_s^{s+h} E\Big[\big(\Gamma(s,x) - \Gamma(r,x)\big)\big(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(z_r),\ u^*_r)\big)\Big]dr \\
& + \int_s^{s+h} E\Big[\Gamma(r,x)(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(z_r),\ u^*_r) \\
& \qquad - f(r,\ \xi^*_{s,r}(x),\ u_r) + f(r,\ \xi^*_{s,r}(x),\ u^*_r))\Big]dr \\
& + \int_s^{s+h} E\Big[\Gamma(r,x)(f(r,\ \xi^*_{0,r}(x_0),\ u_r) - f(r,\xi^*_{0,r}(x_0),\ u^*_r))\Big]dr \\
= & \ I_1(h) + I_2(h) + I_3(h) + I_4(h), \quad \text{say.}
\end{aligned}
$$

Now,

$$
\begin{aligned}
|I_1(h)| &\le K_1 \int_s^{s+h} E\Big[|\Gamma(s,z_r) - \Gamma(s,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]dr \\
&\le K_1 h \sup_{s \le r \le s+h} E\Big[|\Gamma(s,z_r) - \Gamma(s,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big] \\
|I_2(h)| &\le K_2 \int_s^{s+h} E\Big[|\Gamma(s,x) - \Gamma(r,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]dr \\
&\le K_2 h \sup_{s \le r \le s+h} E\Big[|\Gamma(s,z_r) - \Gamma(r,x)|(1 + \|\xi^u(x_0)\|_{s+h}\Big] \\
|I_3(h)| &\le K_3 \int_s^{s+h} E\Big[|\Gamma(r,x)|\ \|x - z_r\|\Big]dr \\
&\le K_3 h \sup_{s \le r \le s+h} E\Big[|\Gamma(r,x)|\ \|x - z.\|_{s+h}\Big].
\end{aligned}
$$

The differences $|\Gamma(s,z_r) - \Gamma(s,x)|$, $|\Gamma(s,x) - \Gamma(r,x)|$ and $\|x - z\|_{s+h}$ are all uniformly bounded in some L^p, $p \ge 1$, and

$$
\begin{aligned}
&\lim_{r \to s} |\Gamma(s,z_r) - \Gamma(s,x)| = 0 \quad \text{a.s.} \\
&\lim_{r \to s} |\Gamma(s,x) - \Gamma(r,x)| = 0 \quad \text{a.s.} \\
&\lim_{h \to 0} \|x - z.\|_{s+h} = 0.
\end{aligned}
$$

Therefore,

$$\lim_{r\to s} \|\Gamma(s,x_r) - \Gamma(s,x)\|_p = 0$$

$$\lim_{r\to s} \|\Gamma(s,x) - \Gamma(r,x)\|_p = 0$$

$$\text{and} \quad \lim_{h\to 0} \|(\|x - z\|_{s+h})\|_p = 0 \quad \text{for some } p.$$

Consequently, $\lim\limits_{h\to 0} h^{-1} I_k(h) = 0$, for $k = 1,2,3$.

The only remaining problem concerns the differentiability of

$$I_4(h) = \int_s^{s+h} E\Big[\Gamma(r,x)(f(r,\ \xi^*_{0,r}(x_0),\ u_r) - f(r,\ \xi^*_{0,r}(x_0),\ u^*_r))\Big] dr.$$

The integrand is almost surely in $L^1([0,T])$ so $\lim\limits_{h\to 0} h^{-1} I_4(h)$ exists for almost every $s \in [0,T]$. However, the set of times $\{s\}$ where the limit may not exist might depend on the control u. Consequently we must restrict the perturbations u of the optimal control u^* to perturbations from a countable dense set of controls. In fact:

1) Because the trajectories are, almost surely, continuous, Y_ρ is countably generated by sets $\{A_{i\rho}\}$, $i = 1,2,\ldots$ for any rational number $\rho \in [0,T]$. Consequently Y_t is countably generated by the sets $\{A_{i\rho}\}$, $r \le t$.

2) Let G_t denote the set of measurable functions from (Ω, Y_t) to $U \subset R^k$. (If $u \in \underline{U}$ then $u(t,w) \in G_t$.) Using the L^1-norm, as in [5], there is a countable dense subset $H_\rho = \{u_{j\rho}\}$ of G_ρ, for rational $\rho \in [0,T]$. If $H_t = \bigcup\limits_{\rho\le t} H_\rho$ then H_t is a countable dense subset of G_t. If $u_{j\rho} \in H_\rho$ then, as a function constant in time, $u_{j\rho}$ can be considered as an admissible control over the time interval $[t,T]$ for $t \ge \rho$.

3) The countable family of perturbations is obtained by considering sets $A_{i\rho} \in Y_t$, functions $u_{j\rho} \in H_t$, where $\rho \le t$, and defining as in 3.1

$$u^*_{j\rho}(s,w) = \begin{cases} u^*(s,w) & \text{if } (s,w) \notin [t,T] \times A_{i\rho} \\ u_{j\rho}(s,w) & \text{if } (s,w) \in [t,T] \times A_{i\rho}. \end{cases}$$

Then for each i,j,ρ

$$\lim_{h\to 0} h^{-1} \int_s^{s+h} E\Big[\Gamma(r,x)(f(r,\ \xi^*_{0,r}(x_0),\ u^*_{j\rho}) - f(r,\ \xi^*_{0,r}(x_0),\ u^*))\Big] dr \tag{5.1}$$

exists and equals

$$E\Big[\Gamma(s,x)(f(s,\ \xi^*_{0,s}(x_0),\ u_{j\rho}) - f(s,\ \xi^*_{0,s}(x_0),\ u^*))I_{A_r}\Big]$$

for almost all $s \in [0,T]$.

Therefore, considering this perturbation we have

$$\lim_{h\to 0} h^{-1}(J(u^*_{j\rho}) - J(u^*)) = E\Big[\Gamma(s,x)(f(s,\ \xi^*_{0,s}(x_0),\ u_{j\rho}) - f(s,\ \xi^*_{0,s}(x_0),\ u^*))I_{A_r}\Big]$$
$$\geq 0 \quad \text{for almost all } s \in [0,T].$$

Consequently there is a set $S \subset [0,T]$ of zero Lebesgue measure such that, if $s \notin S$, the limit in (5.1) exists for all i, j, ρ, and gives

$$E\Big[\Gamma(s,x)(f(s,\ \xi^*_{0,s}(x_0),\ u_{j\rho}) - f(s,\ \xi^*_{0,s}(x_0),\ u^*))I_{A_r}\Big] \geq 0.$$

Using the monotone class theorem, and approximating an arbitrary admissible control $u \in \underline{U}$ we can deduce that if $s \notin S$

$$E\Big[\Gamma(s,x)(f(s,\ \xi^*_{0,s}(x_0),\ u) - f(s,\ \xi^*_{0,s}(x_0),\ u^*))I_A\Big] \geq 0 \quad \text{for any } A \in Y_s. \tag{5.2}$$

Write

$$p_s(x) = E^*\left[c_\xi(\xi^*_{0,T}(x_0))\frac{\partial \xi^*_{s,T}(x)}{\partial x} + c(\xi^*_{0,T}(x_0))\Big(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0))\frac{\partial \xi^*_{s,\sigma}(x)}{\partial x}dv_\sigma\Big) \mid Y_s \vee \{x\}\right]$$

where, as before, $x = \xi^*_{0,s}(x_0)$ and E^* denotes expectation under $P^* = P^{u^*}$. Then $p_s(x)$ is the co-state variable and we have in (5.2) proved the following 'conditional' minimum principle:

THEOREM 5.1. *If $u^* \in \underline{U}$ is an optimal control there is a set $S \subset [0,T]$ of zero Lebesgue measure such that if $s \notin S$*

$$E^*[p_s(x)f(s,x,u^*) \mid Y_s] \leq E^*[p_s(x)f(s,x,u) \mid Y_s] \quad \text{a.s.}$$

That is, the optimal control u^* almost surely minimizes the conditional Hamiltonian and the adjoint variable is $p_s(x)$.

4.6 Conclusions

Using the theory of stochastic flows the effect of a perturbation of an optimal control is explicitly calculated. The only difficulty was to justify its differentiation. The adjoint process is explicitly identified as $p_s(x)$.

THEOREM 6.1. *If f is differentiable in the control variable u, and if the random variable $x = \xi^*_{0,s}(x_0)$ has a conditional density $q_s(x)$ under the measure P^*, then the inequality of Theorem 5.1 implies*

$$\sum_{j=1}^{k}(u_j(s) - u_j^*(s)) \int_{R'} \Gamma(s,x) \frac{\partial f}{\partial u_j}(s,x,u^*) q_s(x) dx \leq 0.$$

This is the result of Bensoussan's paper [1].

The method of this paper can be applied to completely observable systems by initially considering 'stochastic open loop' controls, systems with stochastic constraints and deterministic systems. The adjoint process can be explicitly identified. 'Almost minimum' principles for 'almost optimal' controls can be obtained. Some of these will be discussed in later work.

References

[1] A. Bensoussan, *Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions*. Stochastics, 9(1983), 169–222.

[2] J.M. Bismut, *A generalized formula of Ito and some other properties of stochastic flows*. Zeits. fur Wahrs. 55(1981), 331–350.

[3] J.M. Bismut, *Mecanique Aléatoire*. Lecture Notes in Mathematics 866, Springer-Verlag, 1981.

[4] J.M. Bismut, *Mecanique Aléatoire*. In Ecole d'Eté de Probabilités de Saint-Flour X. Lecture Notes in Mathematics 929, pp. 1–100. Springer-Verlag, 1982.

[5] R.J. Elliott, *The optimal control of a stochastic system*. SIAM Jour. Control and Opt. 15(1977), 756–778.

[6] R.J. Elliott, *Filtering and control for point process observations*. Systems Science Center, University of Maryland. Springer Lecture Notes, to appear.

[7] U.G. Haussmann, *General necessary conditions for optimal control of stochastic systems*. Math. Programming Studies 6(1976), 30–48.

[8] U.G. Haussmann, *A Stochastic Minimum Principle for Optimal Control of Diffusions*. Pitman Research Notes in Mathematics 151, Longman U.K., 1986.

[9] U.G. Haussmann, *The maximum principle for optimal control of diffusions with partial information*. SIAM J. Control and Optimization 25(1987), 341–361.

[10] H. Kunita, *On the decomposition of solutions of stochastic differential equations*. Lecture Notes in Mathematics 851(1980), 213–255.

5 The Conditional Adjoint Process

5.1 Introduction

Using stochastic flows we calculate below the change in the cost due to a 'strong' variation of an optimal control. Differentiating this quantity enables us to identify the adjoint, or co-state variable, and give a partially observed minimum principle. If the drift coefficient is differentiable in the control variable the related result of Bensoussan [2] follows from our theorem. Full details will appear in [1]. The method appears simpler than that employed in Haussman [4].

5.2 Dynamical Equations

Suppose the state of a stochastic system is described by the equation

$$d\xi_t = f(t, \xi_t, u)dt + g(t, \xi_t)dw_t,$$

$$\xi_t \in R^d, \qquad \xi_0 = x_0, \qquad 0 \le t \le T. \tag{2.1}$$

The control variable u will take values in a compact subset U of some Euclidean space R^k. We shall assume

A_1: $x_0 \in R^d$ is given.

A_2: $f : [0,T] \times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t,x), continuously differentiable in x for each (t,u) and

$$(1 + |x|)^{-1}|f(t,x,u)| + |f_x(t,x,u)| \le K_1.$$

A_3: $g : [0,T] \times R^d \to R^d \otimes R^n$ is a matrix valued function, Borel measurable, continuously differentiable in x, and for some K_2:

$$|g(t,x)| + |g_x(t,x)| \le K_2.$$

The observation process is defined by

$$dy_t = h(\xi_t)dt + d\nu_t \tag{2.2}$$

$$y_t \in R^m, \qquad y_0 = 0, \qquad 0 \le t \le T.$$

In (2.1) and (2.2) $w = (w^1, \ldots, w^n)$ and $\nu = (\nu^1, \ldots, \nu^m)$ are independent Brownian notions defined on a probability space (Ω, F, P).

Furthermore, we assume

A_4: $h : R^d \to R^m$ is Borel measurable, continuously differentiable in x and

$$|h(t,x)| + |h_x(t,x)| \leq K_3.$$

REMARK 2.1. These hypotheses can be weakened to those discussed by Haussman [4]. See [1].

Write $\hat{P}$ for the Wiener measure on $C([0,T], R^n)$ and μ for the Wiener measure on $C([0,T], R^m)$.

$$\Omega = C([0,T], R^n) \times C([0,T], R^m)$$

and the coordinate functions in Ω will be denoted (x_t, y_t). Wiener measure P on Ω is

$$P(dx, dy) = \hat{P}(dx)\mu(dy).$$

DEFINITION 2.2. $Y = \{Y_t\}$ *will be the right continuous, complete filtration on* $C([0,T], R^m)$ *generated by*

$$Y_t^0 = \sigma\{y_s : s \leq t\}.$$

The set of admissible control functions $\underline{U}$ *will be the* Y*-predictable functions defined on* $[0,T] \times C([0,T], R^m)$ *with values in* U.

For $u \in \underline{U}$ and $x \in R^d$, $\xi^u_{s,t}(x)$ will denote the strong solution of (2.1) corresponding to u with $\xi^u_{s,s} = x$.

Define

$$Z^u_{s,t}(x) = \exp\left(\int_s^t h(\xi^u_{s,r}(x))' dy_r - \frac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2 dr\right). \tag{2.3}$$

Note a version of Z defined for every trajectory y can be obtained by integrating the stochastic integral in the exponential by parts.

If a new probability measure P^u defined on Ω by putting

$$\frac{dP^u}{dP} = Z^u_{0,T}(x_0),$$

under P^u $(\xi^u_{0,t}(x_0), y_t)$ is a solution of the system (2.1) and (2.2). That is, under P^u, $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion ν such that y_t satisfies (2.2).

Because of hypothesis A_4, for $0 \leq t \leq T$ easy applications of Burkholder's and Gronwall's inequalities show that

$$E[(Z^u_{0,t}(x_0))^p] < \infty \tag{2.4}$$

for all $u \in \underline{U}$ and all p, $1 \leq p < \infty$.

COST 2.3. We shall suppose the cost is purely terminal and equals

$$c(\xi^u_{0,T}(x_0))$$

where c is a bounded, differentiable function. If control $u \in \underline{U}$ is used the expected cost is

$$J(u) = E_u[c(\xi^u_{0,T}(x_0))].$$

With respect to P, under which y_t is a Brownian motion

$$J(u) = E[Z^u_{0,T}(x_0)c(\xi^u_{0,T}(x_0))]. \tag{2.5}$$

A control $u^* \in \underline{U}$ is optimal if

$$J(u^*) \leq J(u)$$

for all $u \in \underline{U}$. We shall suppose there is an optimal control u^*.

5.3 Flows

For $u \in \underline{U}$ and $x \in R^d$ consider the strong solution

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)dr + \int_s^t g(r, \xi^u_{s,r}(x))dw_r. \tag{3.1}$$

We wish to consider the behaviour of $\xi^u_{s,t}(x)$ for each trajectory y of the observation process. In fact the results of Bismut [3] and Kunita [6] extend and show the map

$$\xi^u_{s,t} : R^d \to R^d$$

is, almost surely, a diffeomorphism for each $y \in C([0,T], R^m)$.

Write

$$\|\xi^u(x_0)\|_t = \sup_{0 \le s \le t} |\xi^u_{0,s}(x_0)|.$$

Then, using Gronwall's and Jensen's inequalities, for any p, $1 \le p < \infty$

$$\|\xi^u(x_0)\|^p_T \le C\Big(1 + |x_0|^p + \Big|\int_0^T g(r, \xi^u_{0,r}(x_0))dw_r\Big|^p\Big)$$

almost surely, for some constant C.

Using A_3 and Burkholder's inequality

$$\|\xi^u(x_0)\|_T \in L^p \quad \text{for} \quad 1 \le p < \infty.$$

Suppose u^* is an optimal control, and write

$$\xi^*_{s,t}(\cdot) \quad \text{for} \quad \xi^{u^*}_{s,t}(\cdot).$$

The Jacobian $\frac{\partial \xi^*_{s,t}}{\partial x}$ is the matrix solution C_t of the equation

$$dC_t = f_x(t, \xi^*_{s,t}(x), u^*)C_t dt + \sum_{i=1}^n g_x^{(i)}(t, \xi^*_{s,t}(x))C_t dw^i_t. \tag{3.2}$$

with $C_s = I$.

Here $g^{(i)}$ is the i^{th} column of g and I is the $n \times n$ identity matrix. Writing $\|C\|_T = \sup_{0 \leq s \leq t} |C_s|$ and using Burkholder's, Jensen's and Gronwall's inequalities we see $\|C\|_T \in L^p,\ 1 \leq p < \infty$.

Consider the matrix valued process D defined by

$$D_t = I - \int_s^t D_r f_x(r, \xi^*_{s,r}(x), u^*_r) dr$$
$$- \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \xi^*_{s,r}(x)) dw^i_r + \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \xi^*_{s,r}(x)))^2 dr \tag{3.3}$$

Then as in [5] or [6] $d(D_t C_t) = 0$ and $D_s C_s = I$ so

$$D_t = C_t^{-1} = \left(\frac{\partial \xi^*_{s,t}}{\partial x}\right)^{-1}.$$

Furthermore, $\|D\|_t \in L^p,\ 1 \leq p < \infty$.

Suppose $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i dw^i_r$ is a d-dimensional semimartingale. Bismut [3] shows one can consider the process $\xi^*_{s,t}(z_t)$ and in fact:

$$\xi^*_{s,t}(z_t) = z_s + \int_s^t \Big(f(r, \xi^*_{s,r}(z_r), u^*_r)$$
$$+ \sum_{i=1}^n g_x^{(i)}(r, \xi^*_{s,r}(z_r), u^*_r)\ \frac{\partial \xi^*_{s,r}}{\partial x} H_i$$
$$+ \frac{1}{2} \sum_{i=2}^n \frac{\partial^2 \xi^*_{s,r}}{\partial x^2}(H_i, H_i)\Big) dr$$
$$+ \int_s^t \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) dA_r + \sum_{i=1}^n \int_s^t \Big(g^{(i)}(r, \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \Big) dw^i_r. \tag{3.4}$$

DEFINITION 3.1. *For $s \in [0,T]$, $h > 0$ such that $0 \leq s < s+h \leq T$, for any $\tilde{u} \in \underline{U}$, and $A \in Y_s$ consider a 'strong' variation u of u^* defined by*

$$u(t,w) = \begin{cases} u^*(t,w) & \text{if } (t,w) \notin [s, s+h] \times A \\ \tilde{u}(t,w) & \text{if } (t,w) \in [s, s+h] \times A. \end{cases}$$

THEOREM 3.2. *For any strong variation u of u^* consider the process*

$$z_t = x + \int_s^t \Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr. \tag{3.5}$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

PROOF We shall substitute in (3.4), (noting $H_i = 0$ for all i). Therefore,

$$\begin{aligned}\xi^*_{s,t}(z_t) = x &+ \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r)dr \\ &+ \int_s^t \Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)\Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr \\ &+ \int_s^t g(r, \xi^*_{s,r}(z_r))dw_r.\end{aligned}$$

The solution of (3.1) is unique, so $\xi^*_{s,t}(z_t) = \xi^u_{s,t}(x)$. Note $u(t) = u^*(t)$ if $t > s + h$ so $z_t = z_{s+h}$ if $t > s + h$ and

$$\begin{aligned}\xi^*_{s,t}(z_t) &= \xi^*_{s,t}(z_{s+h}) \\ &= \xi^*_{s+h,t}(\xi^u_{s,s+h}(x)).\end{aligned} \tag{3.6}$$

5.4 The Exponential Density

Consider the $(d+1)$-dimensional system

$$\begin{aligned}\xi^*_{s,t}(x) &= x + \int_s^t f(r, \xi^*_{s,r}(x), u^*_r)dr + \int_s^t g(r, \xi^*_{s,r}(x))dw_r \\ Z^*_{s,t}(x,z) &= z + \int_s^t Z^*_{s,r}(x,z)h(\xi^*_{s,r}(x))'dy_r.\end{aligned} \tag{4.1}$$

That is, we are considering an augmented flow (ξ, Z) in R^{d+1} in which Z^* has a variable initial condition $z \in R$. Note:

$$Z^*_{s,t}(x,z) = zZ^*_{s,t}(x).$$

The map $(x, z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x, z))$ is, almost surely, a diffeomorphism of R^{d+1}. Clearly,

$$\frac{\partial \xi^*_{s,t}}{\partial z} = 0, \qquad \frac{\partial f}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial g}{\partial z} = 0.$$

The Jacobian of this augmented map is represented by the matrix

$$\tilde{C}_t = \begin{pmatrix} \frac{\partial \xi^*_{s,t}}{\partial x} & 0 \\ \frac{\partial Z^*_{s,t}}{\partial x} & \frac{\partial Z^*_{s,t}}{\partial z} \end{pmatrix}.$$

In particular, from (4.1), for $1 \leq i \leq d$

$$\frac{\partial Z^*_{s,t}}{\partial x_i} = \sum_{j=1}^{m} \int_s^t \Big(Z^*_{s,r}(x, z) \sum_{k=1}^{n} \frac{\partial h^j}{\partial \xi_k} \cdot \frac{\partial \xi^*_{k,s,r}}{\partial x_i} + h^j\left(\xi^*_{s,r}(x)\right) \frac{\partial Z^*_{s,r}}{\partial x_i}\Big) dy^j_r. \tag{4.2}$$

We are interested in solutions of (4.1) and (4.2) only when $z = 1$, so as above we write

$$Z^*_{s,t}(x) \quad \text{for} \quad Z^*_{s,t}(x, 1) \quad \text{etc.}$$

LEMMA 4.1.

$$\frac{\partial Z^*_{s,t}}{\partial x} = Z^*_{s,t}(x)\Big(\int_s^t h_x(\xi^*_{s,t}(x)) \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big)$$

where, as in (2.2), $d\nu_t = dy_t - h(\xi^*_{s,t}(x))dt$.

PROOF From (4.2)

$$\frac{\partial Z^*_{s,t}}{\partial x} = \int_s^t \Big(\frac{\partial Z^*_{s,r}}{\partial x} h'(\xi^*_{s,r}(x)) + Z^*_{s,r}(x) h_x(\xi^*_{s,r}(x)) \frac{\partial \xi^*_{s,r}}{\partial x}\Big) dy_r. \tag{4.3}$$

Write

$$L_{s,t}(x) = Z^*_{s,t}(x)\Big(\int_s^t h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big).$$

Then

$$Z^*_{s,t}(x) = 1 + \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r$$

and the product rule gives

$$L_{s,t}(x) = \int_s^t L_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r$$
$$+ \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} dy_r.$$

Consequently, $L_{s,t}(x)$ is also a solution of (4.3), so by uniqueness

$$L_{s,t}(x) = \frac{\partial Z^{*}_{s,t}}{\partial x}.$$

LEMMA 4.2. *If z_t is as defined in (3.5)*

$$Z^{*}_{s,t}(z_t) = Z^{u}_{s,t}(x).$$

PROOF

$$Z^{u}_{s,t}(x) = 1 + \int_s^t Z^{u}_{s,r}(x) h'(\xi^{u}_{s,r}(x)) dy_r. \tag{4.4}$$

Applying (3.4) to $Z^{*}_{s,t}(z_t)$ we see:

$$\begin{aligned} Z^{*}_{s,t}(z_r) &= 1 + \int_s^t Z^{*}_{s,r}(z_r) h'(\xi^{*}_{s,r}(z_r)) dy_r \\ &= 1 + \int_s^t Z^{*}_{s,r}(z_r) h'(\xi^{u}_{s,r}(x)) dy_r \end{aligned}$$

by Theorem 3.2. However, (4.4) has a unique solution so

$$Z^{*}_{s,t}(z_r) = Z^{u}_{s,t}(x).$$

Again, note that for $t > s + h$

$$Z^{*}_{s,t}(z_t) = Z^{*}_{s,t}(z_{s+h}). \tag{4.5}$$

5.5 The Adjoint Process

u^* will be an optimal control and u a perturbation of u^* as in Definition 3.1. Again write

$$x = \xi^{*}_{0,s}(x_0).$$

The minimum cost is

$$J(u^*) = E[Z^*_{0,T}(x_0)c(\xi^*_{0,T}(x_0))]$$
$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(x)c(\xi^*_{s,T}(x))].$$

Also,

$$J(u) = E[Z^*_{0,s}(x_0)Z^u_{s,T}(x)c(\xi^u_{s,T}(x))]$$
$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(z_{s+h})c(\xi^*_{s,T}(z_{s+h}))]$$

by (3.6) and (4.5). Recall $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are differentiable almost surely, with continuous and uniformly integrable derivatives. Consequently, writing

$$\Gamma(s, z_r) = Z^*_{0,s}(x_0)Z^*_{s,T}(z_r)\Big\{c_\xi(\xi^*_{s,T}(z_r))\,\frac{\partial\xi^*_{s,T}}{\partial x}\,(z_r)$$
$$+ c(\xi^*_{s,T}(z_r))\Big(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r))\,\frac{\partial\xi^*_{s,\sigma}}{\partial x}\,(z_r)d\nu_\sigma\Big)\Big\}\Big(\frac{\partial\xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1}$$

for $s \le r \le s+h$, we have

$$J(u) - J(u^*) = E[Z^*_{0,s}(x_0)\{Z^*_{s,t}(z_{s+h})c(\xi^*_{s,t}(z_{s+h})) - Z^*_{s,T}(x)c(\xi^*_{s,T}(x))\}]$$
$$= E\Big[\int_s^{s+h}\Gamma(s, z_r)(f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(x), u^*_r))dr\Big]. \tag{5.1}$$

This formula describes the change in the expected cost arising from the perturbation u of the optimal control. However, $J(u) \ge J(u^*)$ for all $u \in \underline{U}$ so the right hand side of (5.1) is non-negative for all $h > 0$. We wish to divide by $h > 0$ and let $h \to 0$. This requires some careful arguments using the uniform boundedness of the random variables and the monotone class theorem. It can be shown that there is a set $S \subset [0,T]$ of zero Lebesgue measure such that if $s \notin S$

$$E[\Gamma(s,x)(f(s,\xi^*_{0,s}(x_0),u) - f(s,\xi^*_{0,s}(x_0),u^*_s))I_A] \ge 0 \tag{5.2}$$

for any $u \in U$ and $A \in Y_s$.

Details of this argument can be found in [1]. Define

$$p_s(x) = E^*\Big[c_\xi(\xi^*_{0,T}(x_0))\,\frac{\partial \xi^*_{s,T}}{\partial x}(x) + c(\xi^*_{0,T}(x_0))\Big(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0))\,\frac{\partial \xi^*_{s,\sigma}}{\partial x}\,(x)d\nu_\sigma\Big)\Big|\,Y_s \vee \{x\}\Big]$$

where $x = \xi^*_{0,s}(x_0)$ and E^* is the expectation under $P^* = P^{u^*}$.

In (5.2) we have established the following:

THEOREM 5.1. *$p_s(x)$ is the adjoint process for the partially observed optimal control problem. That is, if $u^* \in \underline{U}$ is optimal there is a set $S \subset [0,T]$ of zero Lebesgue measure such that for $s \notin S$*

$$E^*[p_s(x)f(s,x,u^*)\mid Y_s] \geq E^*[p_s(x)f(s,x,u)\mid Y_s] \quad \text{a.s.} \tag{5.3}$$

so the optimal control u^* almost surely minimizes the conditional Hamiltonian.

If $x = \xi^*_{0,s}(x_0)$ has a conditional density $q_s(x)$ under P^*, and if f is differentiable in u, (5.3) implies

$$\sum_{i=1}^k (u_i(s) - u^*_i(s)) \int_{R^d} \Gamma(s,x)\,\frac{\partial f}{\partial u_i}\,(s,x,u^*)q_s(x)dx \geq 0.$$

This is the result of Bensoussan [2].

REFERENCES

1. J. Baras, R.J. Elliott and M. Kohlmann, *The partially observed stochastic minimum principle*. University of Alberta Technical Report, 1987, submitted.

2. A. Bensoussan, *Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions*, Stochastics, 9(1983), 169–222.

3. J.M. Bismut, *A generalized formula of Ito and some other properties of stochastic flows*. Zeits. fur Wahrs. 55(1981), 331–350.

4. U.G. Haussmann, *The maximum principle for optimal control of diffusions with partial information*. S.I.A.M. Jour. Control and Opt. 25(1987), 341–361.

5. N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*. North Holland Publishing Co., Amsterdam, Oxford, New York, 1981.

6. H. Kunita, *The decomposition of solutions of stochastic differential equations*. Lecture Notes in Math., 851(1980), 213-255.

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The focus of this report is on advanced tools for the analysis of nonlinear stochastic control and filtering systems.

In Sections 1 and 2, we present a series of results on the analysis of certain classes of nonlinear filtering problems using comparatively simple bounding techniques. We consider both problems with small noise (large signal to noise ratios) and weakly nonlinear systems. We show that the optimal nonlinear filters can be well approximated by linear filters which are very easy to implement. Moreover, we provide sharp estimates of the degree of sub-optimality involved in using the linear approximating filters.

In Section 3, we consider the problem of managing the estimation of (nonlinear) diffusion process by a system employing several sensors. The essential problem is to "schedule" the use of the sensor to optimize the estimate of a function of the state of the diffusion

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process. The solution is obtained in terms of a system of quasi-variational inequalities in the space of solutions of certain Zakai equations.

In Section 4, we provide a new proof of the minimum principle in stochastic optimal control theory for systems of partially observed diffusions. In Section 5, we provide a concise analysis of the "conditional adjoint process" arising in the stochastic minimum principle for partially observed diffusion processes.

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